AERO 2258A THIN AEROFOIL THEORY Lecture Notes

Author : Hadi Winarto

Two-dimensional, incompressible, inviscid and irrotational flow

This note is prepared as lecture material for the course AERO 2258A Fundamentals of Aerodynamics for the topic of Thin Aerofoil Theory.
It begins with a discussion on the governing equations for 2-dimensional, incompressible and inviscid flow, which is the Laplace equation. Discrete singularities that are the elementary solutions of the Laplace equation are then discussed. It is then followed by a discussion on the linearity property of the Laplace equation, which leads to a discussion on the concept of continuous singularities such as the vortex panel with constant and linear vortex strength per unit length distribution. This is the basic concept behind the panel method, which is explained in some detail, particularly for a second order vortex panel method. Even though the panel method is described in some detail, this is not an article on panel method as such. The concept is introduced so that students will have some familiarity with the concept of continuously distributed singularity, vortex in particular, which is a fundamental concept in Thin Aerofoil Theory (TAT). The basic concept in TAT is that an aerofoil is replaced by a single vortex panel on which there is a continuously distributed vortex singularities, the strength per unit length of which is unknown and needs to be evaluated. A good understanding of this basic concept is not possible without some rudimentary knowledge of a vortex panel. The concept of boundary conditions, that must be satisfied by the sought for solution, is also discussed. The mathematics involved is discussed in some detail, but from an engineering point of view where mathematical knowledge is used as a tool to help in solving engineering problems. Basic mathematical formulas are assumed as given without showing their derivations. The manipulation and application of those mathematical formulas are, however, shown in sufficient details so that students can gain understanding on how the final engineering equations are obtained and not merely given as formulas to be memorized blindly. Finally the engineering application of the derived formulas to calculate the aerodynamic properties of aerofoils is discussed in relatively great details. The application discussed includes aerofoils, which can be represented by a flat plate or a cambered (curved) plate, as well as a control surface, which is represented by a bent flat plate.

Please report any typographical or other errors to the author at the following address
hadi.winarto@rmit.edu.au
Bundoora, Melbourne, 15 March 2004
Last modified 25 May 2004

1. *The governing equations*

The simplest model of airflow is represented by 2-D, incompressible, inviscid and irrotational flow. The governing equations for this type of flow consist of 2 partial differential equations, each representing the conservation of mass or the Continuity Equation, and the irrotationality condition.

Continuity Equation: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

Irrotationality condition: $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0 \quad (2)$$

Two scalar functions (magnitude only, without direction) can be defined so that the two components of velocity (vectors), u and v, in the above equations can be replaced by the equivalent but simpler scalar functions, namely the stream function and the velocity potential function.

Stream Function, ψ, is defined to satisfy the continuity equation as follows

$$u=\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v=-\frac{\partial \psi}{\partial x} \quad (3)$$

Therefore $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)=0$$

It can be seen that the stream function automatically satisfies the continuity equation. Furthermore, the stream function must also satisfy the irrotationality conditions and thus

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)-\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)=\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial x^2}=0 \quad (4)$$

The potential function is defined to automatically satisfy the irrotationality condition as follows

$$u=\frac{\partial \varphi}{\partial x} \qquad \text{and} \qquad v=\frac{\partial \varphi}{\partial y} \quad (5)$$

thus $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right)=0$$

Since the potential function must also satisfy the continuity equation, therefore

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y}\right)=\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0 \quad (6)$$

It can be seen that the two first order partial differential equations in 2 unknowns, namely equations (1) and (2), can be replaced by a single second order elliptic partial differential equation, namely the Laplace equation either in terms of stream function (equation (4)) or in terms of the potential function (equation (6)).
Furthermore, it can be shown that the stream function and the potential function are harmonic conjugate of each other, and thus we can define a complex potential function as follows

$$\Phi(z)=\varphi(x,y)+i.\psi(x,y) \quad (7)$$

where z is the complex variable

$$z = x + i.y \tag{8}$$

and i is the imaginary number

$$i = \sqrt{-1} \tag{9}$$

Equations (4) and (6) can be combined as follows

$$\nabla^2\Phi(z) = \nabla^2\left(\varphi(x,y) + i.\psi(x,y)\right) = \nabla^2\varphi(x,y) + i.\nabla^2\psi(x,y) = 0 \tag{10}$$

2. *Solution of the Laplace equation*

From the theory of complex variable, it is known that the solution of the Laplace Equation in terms of a complex function is any complex analytic function. We will not discuss this further except to note that the theory helps us in obtaining a very large number of elementary solutions of the Laplace equation. Among the very large number of elementary solutions, also known as singularities, there are four (4) that are especially useful in the study of aerodynamics, namely the source, sink, doublet and vortex singularities. Source and sink are actually the same type of singularity except that they have opposite signs for their strength.
The problem of a uniform airflow, which is disturbed by the presence of an aerofoil located within the flow field, is modelled by assuming that the disturbance can be represented mathematically by the singularities, solutions of the Laplace equation. The Laplace equation is said to be linear, meaning that a linear combination of some simpler or elementary solutions is also a solution.
Let us now have a quick look at the elementary solutions

Source / sink : singularity located at (x_0, y_0) with a strength of σ

Stream function induced at (x,y):
$$\psi(x,y) = \pm\frac{\sigma}{2\pi}.\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) \tag{11}$$

Potential function induced at (x,y):
$$\varphi(x,y) = \pm\frac{\sigma}{2\pi}.\ln\left(\left(x-x_0\right)^2 + \left(y-y_0\right)^2\right)^{\frac{1}{2}} \tag{12}$$

Doublet : singularity located at (x_0, y_0) with a strength of μ

Stream function induced at (x,y):
$$\psi(x,y) = \frac{\mu}{2\pi}.\frac{y-y_0}{\left(x-x_0\right)^2 + \left(y-y_0\right)^2} \tag{13}$$

Potential function induced at (x,y):
$$\varphi(x,y) = -\frac{\mu}{2\pi}.\frac{x-x_0}{\left(x-x_0\right)^2 + \left(y-y_0\right)^2} \tag{14}$$

Vortex : singularity located at $\left(x_0, y_0\right)$ with a strength of Γ

Stream function induced at (x,y): $\psi\left(x, y\right) = \frac{\Gamma}{2\pi}.\ln\left(\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right)^{\frac{1}{2}}$ (15)

Potential function induced at (x,y): $\varphi\left(x, y\right) = -\frac{\Gamma}{2\pi}.\tan^{-1}\left(\frac{y - y_0}{x - x_0}\right)$ (16)

<u>The velocity components</u> induced at (x,y) by the presence of the singularity at $\left(x_0, y_0\right)$ can be evaluated as follows.

x-component of velocity: $u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$ (17)

y-component of velocity: $v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$ (18)

Source / sink : singularity located at $\left(x_0, y_0\right)$ with a strength of σ

x-component: $u\left(x, y\right) = \pm\frac{\sigma}{2\pi}.\frac{x - x_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$ (19)

y-component: $v\left(x, y\right) = \pm\frac{\sigma}{2\pi}.\frac{y - y_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$ (20)

Doublet : singularity located at $\left(x_0, y_0\right)$ with a strength of μ

x-component: $u\left(x, y\right) = \frac{\mu}{2\pi}.\frac{\left(x - x_0\right)^2 - \left(y - y_0\right)^2}{\left(\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right)^2}$ (21)

y-component: $v\left(x, y\right) = \frac{\mu}{2\pi}.\frac{2\left(x - x_0\right)\left(y - y_0\right)}{\left(\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right)^2}$ (22)

Vortex : singularity located at $\left(x_0, y_0\right)$ with a strength of Γ

x-component: $u\left(x, y\right) = \frac{\Gamma}{2\pi}.\frac{\left(y - y_0\right)}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$ (23)

y-component: $v\left(x, y\right) = -\frac{\Gamma}{2\pi}.\frac{\left(x - x_0\right)}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$ (24)

The mathematical expressions for the stream and potential functions of a uniform flow, which is inclined at an angle α to the horizontal axis x, are given as follows

Stream function value induced at (x,y): $$\psi(x,y)=V_\infty.(y.\cos\alpha - x.\sin\alpha) \qquad (25)$$

Potential function value induced at (x,y) $$\varphi(x,y)=V_\infty.(x.\cos\alpha + y.\sin\alpha) \qquad (26)$$

The x- component of velocity is: $$u(x,y)=\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y}=V_\infty.\cos\alpha \qquad (27)$$

The y- component of velocity is: $$v(x,y)=\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}=V_\infty.\sin\alpha \qquad (28)$$

The quantity V_∞ is the speed of the undisturbed air or the free-stream air.

3. *Linearity property of the Laplace equation*

It was previously mentioned that the Laplace Equation is a linear second order partial differential equation. The linearity property of the equation implies that any linear combination of some elementary solutions is also a solution. It was mentioned also that the problem of air flow over an aerofoil (or in American terminology: airfoil) is modelled as a uniform air flow which is disturbed by the presence of the aerofoil. We shall now have a look whether the aerofoil can be represented by a single source, or a single doublet or a single vortex. For simplicity it will be assumed that the value of the angle α is zero.

Linear combination of uniform flow and a source:

$$\psi(x,y)=V_\infty.y+\frac{\sigma}{2\pi}.\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) \qquad (29)$$

Linear combination of uniform flow and a doublet:

$$\psi(x,y)=V_\infty.y+\frac{\mu}{2\pi}.\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} \qquad (30)$$

Linear combination of uniform flow and a vortex:

$$\psi(x,y)=V_\infty.y+\frac{\Gamma}{2\pi}.\ln\left((x-x_0)^2+(y-y_0)^2\right)^{\frac{1}{2}} \qquad (31)$$

The flow pattern of a flow field is defined by the streamlines of the flow.
For a steady flow, i.e. one that does not change with time, if we release a particle at a point in the flow field and then follow the motion of the particle as it is swept downstream, the path of the particle motion is known as a streamline. Another particle released at another point would describe another streamline. By drawing a number of

streamlines, we can get a picture of the shape of the flow field. For an unsteady flow or turbulent flow, obviously the shape of the streamlines will change continuously with time. However, for a steady flow the pattern of the flow field is constant. Another property of the streamline is the fact that at any given point on the streamline, the direction of the velocity vector is tangent to the streamline at the given point. Therefore no flow can cross a streamline. Furthermore, the stream function value at any point on the streamline is a constant. In other words, we can define a streamline as being the locus of points within the flow field where the values of the stream function at all points on the streamline are the same. Thus a streamline can also be called a constant stream function curve

Now let us consider what sort of a flow field we get from the combination of a uniform flow, which is disturbed by a doublet.

To simplify the discussion let us assume that the doublet is located at the origin of the system of axes, or at $x_0 = 0 \, and \, y_0 = 0$. Furthermore, in order to get a meaningful result it is assumed that the sign of the doublet strength is negative. With those assumptions equation (30) can now be written as follows

$$\psi(x, y) = V_\infty . y - \frac{\mu}{2\pi} . \frac{y}{x^2 + y^2} = V_\infty . y . \left(1 - \frac{R^2}{x^2 + y^2}\right) \tag{32}$$

where $$R^2 = \frac{\mu}{2\pi V_\infty} \text{ is a positive constant} \tag{33}$$

Let us now have a look at a particular streamline with a value of stream function of zero. If $\psi = 0$ then equation (32) is simplified to become

$$y . \left(1 - \frac{R^2}{x^2 + y^2}\right) = 0 \tag{34}$$

There are 2 solutions to the above equation, namely

$$y = 0 \tag{35}$$

and also

$$1 - \frac{R^2}{x^2 + y^2} = 0$$

The above equation can be simplified further as follows

$$x^2 + y^2 = R^2 \tag{36}$$

Equation (36) is the equation of a circle whose centre is located at the origin and its radius is R. Since fluid can not cross a streamline, therefore a streamline may be replaced by an impermeable wall. Therefore equation (32) actually represents the flow field of a uniform flow which is flowing over a circular cylinder with radius R. The streamline with a stream function value of $\psi = 0$, which includes the x-axis or equation (35) and the circle given by equation (36), is known as the dividing streamline. This streamline divides the flow field into 2 parts, namely one that flows

over the upper part of the circle and another which flows over the lower part of the same circle or circular cylinder.
All the other streamlines, which describe the flow over the upper part of the circle, can be obtained by solving the following equation

$$y.\left(1-\frac{R^2}{x^2+y^2}\right)=\frac{\psi}{V_\infty} \tag{37}$$

where $\dfrac{\psi}{V_\infty}$ is a constant and $\dfrac{\psi}{V_\infty} > 0$.
Similarly, all the other streamlines that describe the flow over the lower part of the circle are solutions of equation (37) where $\dfrac{\psi}{V_\infty} < 0$.
If we evaluate the coordinates (x, y) of a large number of points within the flow field for a particular value of ψ / V_∞, and all the neighbouring points are connected to each other by short straight lines, then we will get the shape of the streamline for that particular value of ψ / V_∞. The flow pattern that we wish to analyse is then given visually as a collection of streamlines for various values of ψ / V_∞.
A doublet represents the disturbance of a circular cylinder immersed in a uniform flow, whereas a source or a vortex represents another type of disturbance.. However none of those would represent the disturbance of an aerofoil. The flow pattern of a uniform wind being disturbed by a source or a vortex will not be discussed here, but they can be readily found in most textbooks on aerodynamics.
It is important to note here that whilst the flow around a circular cylinder is not particularly important in aerodynamics, however we can employ the theory of complex variables to transform the flow field around a circular cylinder into that of the flow around an aerofoil. In the simplest case, the transform function or mapping function is assumed known and the shape of the aerofoil is obtained as a result. Perhaps the most well known mapping function is the Joukowski's transformation, which really is a special form of the more general Karman-Trefftz mapping function. If we want to obtain the flow field around any arbitrary shape aerofoil, then we will need to evaluate the mapping function. This is a far more difficult problem than for Joukowski or Karman-Trefftz mapping, and will not be discussed here. It is sufficient to note that one of the possible methods to use is the Laurent Series transformation. The use of complex variable transformation is known as the **conformal mapping** method (see appendix 4).

4. *Discrete and Continuous Singularities*

Let us now return to our original statement, which is that a more complex flow can be obtained by adding together a number of simpler solutions and see if this can be used to obtain a more direct solution to the problem of evaluating the flow field of a uniform wind which is disturbed by an arbitrary shape aerofoil. This more direct approach is known as the panel method. Here we will discuss the basics of the 2-D panel method only, but the same approach may be applied to 3-D problems. This is in

contrast to the conformal mapping method, which is only applicable for 2-D problems.
The stream function value at any point (x, y) in a uniform flow, which is disturbed by 2 vortices, each being located at (x_1, y_1) and (x_2, y_2) respectively, is given by the following expression

$$\psi(x,y) = V_\infty . y + \frac{\Gamma_1}{2\pi}.\ln\left((x-x_1)^2 + (y-y_1)^2\right)^{\frac{1}{2}} + \frac{\Gamma_2}{2\pi}.\ln\left((x-x_2)^2 + (y-y_2)^2\right)^{\frac{1}{2}} \qquad (38)$$

If there are N, rather than 2, vortices then we have

$$\psi(x,y) = V_\infty . y + \sum_{n=1}^{N} \frac{\Gamma_n}{2\pi}.\ln\left((x-x_n)^2 + (y-y_n)^2\right)^{\frac{1}{2}} \qquad (39)$$

Let us now consider the situation where there are 2 points, P and Q, with coordinates of (XP, YP) and (XQ, YQ). The straight line from P to Q is divided up into a large number of equal length interval of $\Delta s = PQ/N$, where PQ is the distance from P to Q. Let us now imagine that at the mid point of each small interval there is a vortex of strength G, which is the same for all intervals. The stream function value at any point (x, y) for this case is given by

$$\psi(x,y) = V_\infty . y + \sum_{n=1}^{N} \frac{G}{2\pi}.\ln\left((x-x_n)^2 + (y-y_n)^2\right)^{\frac{1}{2}} \qquad (40)$$

It should be noted that even though N can be made to be very large, approaching infinity, we shall impose the condition that the total strength of the vortices is always the same regardless of the actual value of N, and this total vortex strength is Γ where

$$\Gamma = N.G$$

Let us denote the mid point n as being the point S_n, such that the coordinates of the point S_n is (XS_n, YS_n) where

$$\Delta X = (XQ - XP) / N = \Delta XPQ / N$$

$$\Delta Y = (YQ - YP) / N = \Delta YPQ / N$$

$$\Delta s = PQ / N = \sqrt{\Delta XPQ^2 + \Delta YPQ^2} \;/ N$$

$$XS_n = XP + \left(n - \frac{1}{2}\right).\frac{\Delta XPQ}{PQ}.\Delta s = XP + \Delta XPS_n$$

$$YS_n = YP + \left(n - \frac{1}{2}\right).\frac{\Delta YPQ}{PQ}.\Delta s = YP + \Delta YPS_n$$

G is the strength of the discrete vortex located at the midpoint of an elemental length of Δs. If N is chosen to be sufficiently large, then Δs is sufficiently small to be replaced by a continuous differential, i.e. $\Delta s \cong ds$. Furthermore we shall assume that the strength of the vortex, G, is distributed evenly along ds and hence

$$G = \frac{\Gamma}{N} = \gamma . \frac{PQ}{N} = \gamma . \Delta s \cong \gamma . ds$$

Here γ is the strength per unit length of the continuous vortex sheet ds. Since G is the same for all elemental lengths, ds, therefore γ is also the same for all ds along the vortex sheet or panel PQ.

Equation (40) can now be written as follows

$$\psi\left(XT, YT\right) = V_{\infty}.YT + \sum_{n=1}^{N} \frac{\gamma.\Delta s}{2\pi}.\ln\left(\left(XT - XS_n\right)^2 + \left(YT - YS_n\right)^2\right)^{\frac{1}{2}} \qquad (41)$$

where (XT, YT) are the coordinates of the point T, at which the value of the stream function is to be calculated.
Now it is to be noted that

$$XT - XS_n = XT - XP - \Delta XPS_n = \Delta XPT - \Delta XPS_n$$

$$YT - YS_n = YT - YP - \Delta YPS_n = \Delta YPT - \Delta YPS_n$$

Therefore,

$$S_nT^2 = \left(XT - XS_n\right)^2 + \left(YT - YS_n\right)^2 = PT^2 - 2.\left(\Delta XPT.\Delta XPS_n + \Delta YPT.\Delta YPS_n\right) + PS_n^2$$

where it is defined that

$$PT^2 = \Delta XPT^2 + \Delta YPT^2$$

$$PS_n^2 = \Delta XPS_n^2 + \Delta YPS_n^2$$

Equation (41) can now be written as follows

$$\psi\left(XT, YT\right) = V_{\infty}.YT + \sum_{n=1}^{N} \frac{\gamma.\Delta s}{2\pi}.\ln S_nT = V_{\infty}.YT + \sum_{n=1}^{N} \frac{\gamma.\ln S_nT}{2\pi}.\Delta s$$

Taking the limit of N approaching infinity, the above summation can be replaced by the following integral

$$\psi\left(XT, YT\right) = V_{\infty}.YT + \int_0^{PQ} \frac{\gamma}{2\pi} \ln r(s).ds \qquad (42)$$

$$r^2 = PT^2 - 2.\left(\Delta XPT.\Delta XPS + \Delta YPT.\Delta YPS\ \right) + s^2$$

where r(s) is the distance from point S, anywhere along the line PQ, to the point T, and s is the distance from point P to point S.
It can also be observed that

$$\Delta XPS = \frac{s}{PQ}.\Delta XPQ \qquad and \qquad \Delta YPS = \frac{s}{PQ}.\Delta YPQ$$

Therefore

$$r^2 = PT^2 - 2.PR.s + s^2 = TR^2 + \left(PR - s\right)^2 \tag{43}$$

$$PR = \left(\Delta XPT.\Delta XPQ + \Delta YPT.\Delta YPQ\right)/\,PQ \tag{44}$$

$$TR = \sqrt{PT^2 - PR^2} \tag{45}$$

Since γ is a constant, therefore equation (42) can be simplified to become

$$\psi\left(XT, YT\right) = V_{\infty}.YT + \frac{\gamma}{2\pi}\int_0^{PQ} \ln r(s).ds \tag{46}$$

It an be shown (see appendix 1) that

$$\int_P^Q \ln r\left(s\right).ds = \left(PQ - PR\right).\ln QT + PR.\ln PT - PQ + TR.APTQ \tag{47}$$

where

$$APTQ = \tan^{-1}\left(\frac{PQ.TR}{PT^2 - PQ.PR}\right) \tag{48}$$

It should be noted that APTQ is the angle subtended by the lines PT and TQ, or the visible angle of PQ seen from T.

Equation (42) is the value of stream function at point T, which is immersed in the flow field of a uniform flow with a free stream velocity of V_{∞}, and is influenced by the presence of a continuously distributed vortex. The vorticity is distributed on a line PQ, which is actually the intersection of rectangle with the x-y plane or the plane of the paper. The rectangle is perpendicular to the x-y plane and is infinitely long in the z-direction. It is referred to as a panel and because vorticity is distributed on it, therefore, it is called a vortex panel. The vortex strength per unit length distribution along PQ is $\gamma(s)$, which is a function of the variable length representing the distance from P to a point S located anywhere between P and Q. Generally speaking the functional form of $\gamma(s)$ is unknown and represents the problem that has to be solved.

The above discussion shows that the Laplace equation can have a discrete solution as well as a continuous solution. In terms of the stream function equation (15) represents the solution of the stream function value at any arbitrary point (*x, y*) induced by a discrete vortex with strength of Γ and located at $\left(x_0, y_0\right)$. The solution of the same problem in terms of the potential function is given by equation (16), whereas for the velocity components *u* and *v* the solutions are equations (23) and (24) respectively.

If the discrete vortex is replaced by a distributed vortex along panel PQ, with vorticity strength per unit length of $\gamma\left(s\right)$, the solutions are given as follows

$$\psi\left(x, y\right)=\frac{1}{2\pi}\int_P^Q \gamma\left(s\right).\ln r\left(s\right).ds \tag{49}$$

$$\varphi\left(x, y\right)=-\frac{1}{2\pi}\int_P^Q \gamma\left(s\right).\theta\left(s\right).ds \tag{50}$$

$$u\left(x, y\right)=\frac{1}{2\pi}.\int_P^Q \gamma\left(s\right)\frac{\sin\theta\left(s\right)}{r\left(s\right)}.ds \tag{51}$$

$$v\left(x, y\right)=-\frac{1}{2\pi}\int_P^Q \gamma\left(s\right).\frac{\cos\theta\left(s\right)}{r\left(s\right)}.ds \tag{52}$$

where

$$s=\left(\left(XS-XP\right)^2+\left(YS-YP\right)^2\right)^{\frac{1}{2}} \tag{53}$$

$$r\left(s\right)=\left(\left(x-XS\right)^2+\left(y-YS\right)^2\right)^{\frac{1}{2}} \tag{54}$$

$$\theta\left(s\right)=\tan^{-1}\left(\frac{y-YS}{x-XS}\right) \tag{55}$$

It should be remembered that (*XS, YS*) are the coordinates of the point S, which is located anywhere along the panel PQ. The coordinates of points P and Q are (*XP, YP*) and (*XQ, YQ*) respectively.

Similar expressions can also be derived for the case where the singularity is source and sink or doublet, rather than vortex. The final results are given below

Source and Sink:

$$\psi\left(x, y\right)=\pm\frac{1}{2\pi}\int_P^Q \sigma\left(s\right).\theta\left(s\right).ds \tag{56}$$

$$\varphi(x,y)=\pm\frac{1}{2\pi}\int_P^Q \sigma(s).\ln r(s).ds \tag{57}$$

$$u(x,y)=\pm\frac{1}{2\pi}\int_P^Q \sigma(s).\frac{\cos\theta(s)}{r(s)}.ds \tag{58}$$

$$v(x,y)=\pm\frac{1}{2\pi}\int_P^Q \sigma(s).\frac{\sin\theta(s)}{r(s)}.ds \tag{59}$$

Doublet :

$$\psi(x,y)=\frac{1}{2\pi}\int_P^Q \mu(s).\frac{\sin\theta(s)}{r(s)}.ds \tag{60}$$

$$\varphi(x,y)=-\frac{1}{2\pi}\int_P^Q \mu(s).\frac{\cos\theta(s)}{r(s)}.ds \tag{61}$$

$$u(x,y)=\frac{1}{2\pi}\int_P^Q \mu(s).\frac{\cos 2\theta(s)}{r^2(s)}.ds \tag{62}$$

$$v(x,y)=\frac{1}{2\pi}\int_P^Q \mu(s).\frac{\sin 2\theta(s)}{r^2(s)}.ds \tag{63}$$

It should be noted that $\sigma(s)$ and $\mu(s)$ are the strength per unit length of the source and doublet distribution respectively.

It can be seen clearly that the integrals involved in the above equations are very complex and can't be solved unless the strength per unit length distribution of the chosen singularity (source, doublet or vortex) is given. Furthermore, even for the simplest case where the strength per unit length distribution is just a constant it is already quite difficult to evaluate the explicit expression for the integral. Another difficulty is the geometry of the singularity panel. If the panel is not straight but curved instead, then it is impossible to obtain an analytical solution for the integral. On the other hand we know that wings or aerofoils are not flat panels, but are highly curved. In the panel method this problem is overcome by replacing the continuously smooth curve of the aerofoil with an approximate curve consisting of a large number of panels or straight lines connecting adjacent points on the surface of the aerofoil. The overall effect of the whole aerofoil on the value of stream function, or any of the other functions that is chosen, at a point is then obtained as the sum of the effects of all panels representing the aerofoil. Since each panel is a straight line therefore it is possible to derive the expression for the required integral, provided that the strength per unit length distribution of the singularity is kept simple. If the distribution is assumed constant along the panel, then the method is known as the first order panel method, since the distribution only requires the knowledge of one unknown constant. A much better approximating distribution is given by a linear function of s, which

involves two unknown constants, hence it is referred to as a second order panel method. Below we will show the derivation of the expression for the integral on the right hand side of equation (49) for a second order panel method.
The vortex strength per unit length linear distribution is given by the following

$$\gamma(s) = \gamma P + \frac{\gamma Q - \gamma P}{PQ}.s \tag{64}$$

where γP and γQ are the values of $\gamma(s)$ at points P and Q respectively, and represent the two unknown constants of the second order panel method.
The stream function value at (x, y) for this case is thus given by

$$\psi(x, y) = \frac{1}{2\pi}\int_P^Q \left(\gamma P + \frac{\gamma Q - \gamma P}{PQ} s \right).\ln r(s).ds \tag{65}$$

The above equation can be rewritten as follows

$$\psi(x, y) = CI_1.\gamma P + CI_2.(\gamma Q - \gamma P) = (CI_1 - CI_2).\gamma P + CI_2.\gamma Q \tag{66}$$

$$CI_1 = \int_P^Q \ln r(s).ds \tag{67}$$

$$PQ.CI_2 = \int_P^Q s.\ln r(s).ds \tag{68}$$

The expression for the integral in equation (67) has already been worked out in appendix 1 with the folowing result

$$CI_1 = (PQ - PR).\ln QT + PR.\ln PT - PQ + TR.APTQ \tag{69}$$

Details for the evaluation of the second integral are given in appendix (2) the result of which is

$$CI_2 = \tfrac{1}{2}\left[\left(\frac{PQ^2 - (PR^2 - TR^2)}{PQ} \right) \ln QT + \left(\frac{PR^2 - TR^2}{PQ} \right) \ln PT - (PQ + PR) \right] + \frac{PR.TR.APTQ}{PQ} \tag{70}$$

Our discussion so far only deals with the solution of the Laplace equation in general, whereas the real problem that we want to solve is how to analyse the aerodynamic properties of a 2-dimensional wing or an aerofoil. The mathematical model for our simplified problem, i.e. confined to inviscid, incompressible flows only, is indeed the Laplace equation, but we have not discussed about the boundary conditions that must be satisfied. To obtain a unique solution we need to specify what boundary conditions the solution must satisfy.

5. *The boundary conditions*

The real question we want to find the answer for here is that if the shape of a wing or an aerofoil is given, and the wing is immersed in a given airflow, what is the lift and drag acting on the aerofoil as a result of its interaction with the flowing air. It should be remembered that in a Galilean transformation, it makes no difference if the body is stationary and the air is moving (model aircraft in a wind tunnel) or whether the body is moving in a stagnant atmosphere (aircraft moving in the atmosphere).
Drag and lift are the two orthogonal components of the resultant force acting on the aerofoil, which is the summation of all the pressure acting on the surface of the aerofoil. From Bernoulli equation we know that the total pressure in an isenthalpic flow is constant, hence the static pressure decreases if the dynamic pressure increases and vice versa. The dynamic pressure is of course half multiplied by air density multiplied by velocity squared. Thus pressure is directly related to the fluid velocity squared.

Imagine an airflow moving uniformly from the left to the right. Everywhere within the flow field the velocity is the same, i.e. the free stream velocity, thus the static pressure is also the same with a value of free stream static pressure. Now imagine that suddenly an aerofoil is inserted into the airflow. Obviously the presence of the aerofoil would disturb the uniform airflow, and the velocity field would change from the previously uniform value, at least in the vicinity of the aerofoil surface.
On the other hand we have also seen that the singularities, which are the elementary solutions of the Laplace equation, also has the effect of disturbing a uniform flow of fluid, at least in the vicinity of the location of the singularity. From this observation we can make the conclusion that perhaps the airflow around an aerofoil can be simulated by placing singularities on the aerofoil's surface, or on a surface that can be assumed to be representative of the aerofoil surface.

It is an observed fact that air can't penetrate into the inside of the aerofoil, the surface of which is made out of solid. Since the surface of the aerofoil represents part of the boundary of the airflow, therefore the requirement that air can't penetrate into the aerofoil's surface is called a boundary condition. The aerofoil's surface is called the inner boundary since it represents the boundary of the inner part of the flow. The other boundary is the outer boundary, which ideally is infinitely far away from the aerofoil's surface, but from a practical point of view may be defined as being sufficiently far away from the surface. At the outer boundary the flow is undisturbed by the presence of the aerofoil, therefore in our simulated flow the Laplacian singularities placed on the surface also must not disturb the free stream uniform flow far away from the singularities. This requirement is always automatically satisfied by all of the singularities, namely source/sink, doublet and vortex. Therefore, it is only the inner boundary condition that must be satisfied in our simulated flow, i.e. that the flow must not penetrate the aerofoil's surface. This requirement can be expressed in either one of at least two different ways.

Firstly, it may be stated that the surface of the aerofoil is part of the dividing streamline that splits the airflow into an upper half and a lower half of the flow field. Since a streamline is a curve on which the value of the stream function is constant, this type of boundary condition is called the Dirichlet Condition where the value of the function (the stream function) is defined (as being of a constant value) at the boundary of the flow field.

The other way of specifying the boundary condition is that the normal component of the flow velocity at the boundary must be zero, since otherwise it would imply that air is allowed to penetrate into the aerofoil's surface. Since flow velocity is the derivative of the function (i.e. the stream function) this is the same as specifying the values of the function's derivative at the boundary and is known as the Neuman Condition.

Generally speaking, in some problems the function values are specified at some parts of the boundary, while at the rest of the boundary the derivative values are specified. This third type of boundary condition is known as the Mixed Boundary Condition, which is also known as the Robin Condition.

Now we can begin to describe how our problem is to be simulated using a second order vortex panel method.

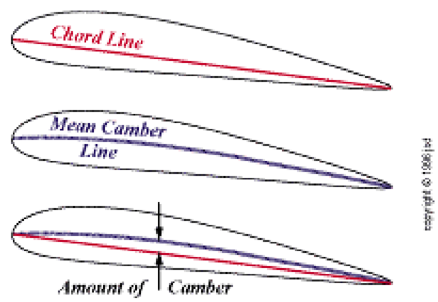


Figure 1. Definition of parts of an aerofoil (from Georgia Institute of Technology web site http://www.adl.gatech.edu/classes/lowspdaero/lospd5/lospd5.html)

The undisturbed flow is represented as a uniform flow the velocity vector of which is inclined at an angle of α relative to the horizontal or x-axis. The aerofoil is fixed in space and is so located such that its chord line is along the x-axis, with the nose being at the origin and the tail being to the right of the nose. The aerofoil's surface is represented by a large but finite number of points on it. The smoothly continuous curved surface is approximated by a piecewise linear segmented continuous surface consisting of small segments of straight lines or panels. Obviously the approximation gets better as the number of points on the aerofoil's surface is increased. However, this has the implication of increasing amount of computational work to be done.

Since in reality all wings have tails with finite thickness, it will be assumed here that the aerofoil has a blunt trailing edge, with distinct lower and upper tail points. The lower trailing edge (tail) point is identified as the point P_1 and the upper tail point is the point P_N, where N is the number of points representing the aerofoil's surface. The number of panels is obviously (N-1). Indexation of all surface points is done in a clockwise direction, thus the point next to and to the left of the first point is identified as the point P_2 etc. The n^{th} panel is the line connecting point P_n to P_{n+1}, which is also identified as the point Q_n. Thus the n^{th} panel is also referred to as the panel PQ_n.

In the following discussion we will describe a solution based on the method that utilizes the Dirichlet boundary conditions. Thus the solution obtained will be in terms of the stream function.

The stream function value at a point T with coordinates (XT,YT) due to T being immersed in a uniform flow, which is inclined at an angle of α to x-axis, is given as follows

$$\psi_{uw}\left(XT,YT\right)=V_{\infty}.\left(-XT.\sin\alpha+YT.\cos\alpha\right) \tag{71}$$

The stream function value at T induced by the vortex panel PQ_n is given by equation (65), where the vortex strength per unit length distribution is assumed given by a linear distribution, i.e. we shall use a second order method. Thus along the panel PQ_n we have the following distribution

$$\gamma_n\left(s\right)=\gamma P_n+\frac{\gamma P_{n+1}-\gamma P_n}{PQ_n}.s \tag{72}$$

For this case equation (65) can be rewritten as follows

$$\psi PQ_n=\tfrac{1}{2\pi}\int_{P_n}^{P_{n+1}}\left(\gamma P_n+\frac{\gamma P_{n+1}-\gamma P_n}{PQ_n}.s\right).\ln r\left(s\right).ds \tag{73}$$

$$\begin{aligned}\psi PQ_n&=\tfrac{1}{2\pi}\left(CI_{1n}.\gamma P_n+CI_{2n}.\left(\gamma P_{n+1}-\gamma P_n\right)\right)\\ \psi PQ_n&=\tfrac{1}{2\pi}\left(\left(CI_{1n}-CI_{2n}\right).\gamma P_n+CI_{2n}.\gamma P_{n+1}\right)\end{aligned} \tag{74}$$

The expressions for CI_{1n} and CI_{2n} are given by equations (69) and (70) as follows

$$CI_{1n}=\left(PQ_n-PR_n\right).\ln PT_{n+1}+PR_n.\ln PT_n-PQ_n+TR_n.APTQ_n \tag{75}$$

$$\begin{aligned}CI_{2n}=&\tfrac{1}{2}\left[\left(\frac{PQ_n^2-\left(PR_n^2-TR_n^2\right)}{PQ_n}\right)\ln PT_{n+1}+\left(\frac{PR_n^2-TR_n^2}{PQ_n}\right)\ln PT_n-\left(PQ_n+PR_n\right)\right]\\ &+\frac{PR_n.TR_n.APTQ_n}{PQ_n}\end{aligned} \tag{76}$$

The stream function value at T induced by all the vortex panels making up the complete aerofoil shape is given by

$$\psi_{vs}\left(XT,YT\right)=\sum_{n=1}^{N-1}\psi PQ_n=\tfrac{1}{2\pi}\sum_{n=1}^{N-1}\left(\left(CI_{1n}-CI_{2n}\right).\gamma P_n+CI_{2n}.\gamma P_{n+1}\right) \tag{77}$$

If it is now defined that

$$C_{T,1} = CI_{11} - CI_{21}$$

$$C_{T,n} = CI_{1n} - CI_{2n} + CI_{2n-1} \quad for\, n = 2,3,...,N-1 \tag{78}$$

$$C_{T,N} = CI_{2N-1}$$

then equation (77) can be rewritten more compactly as follows

$$\psi_{vs}\left(XT,YT\right) = \tfrac{1}{2\pi}\sum_{n=1}^{N} C_{T,n}.\gamma P_n \tag{79}$$

The value of the stream function at point T is the sum of the stream function due to being immersed in the uniform wind plus the stream function induced by all the vortex panels making up the surface of the aerofoil

$$\psi_T = \psi_{uw}\left(XT,YT\right) + \psi_{vs}\left(XT,YT\right)$$

$$\psi_T = V_\infty.\left(-XT.\sin\alpha + YT.\cos\alpha\right) + \tfrac{1}{2\pi}\sum_{n=1}^{N} C_{T,n}.\gamma P_n \tag{80}$$

The above equation can be rewritten as follows

$$\sum_{n=1}^{N} C_{T,n}.\gamma P_n - 2\pi.\psi_T = 2\pi.V_\infty.\left(XT.\sin\alpha - YT.\cos\alpha\right) \tag{81}$$

If the coordinates of all points on the aerofoil's surface $\left(XP_n, YP_n\right)$ and the coordinates of point T, i.e. $\left(XT,YT\right)$, are given then all the influence coefficients, CI_{1n} and CI_{2n} hence $C_{T,n}$ for all values of n from 1 to N can be evaluated.

Now let us define a dummy value of γ_{N+1} as follows

$$\gamma P_{N+1} = \psi_T \tag{82}$$

and also a dummy coefficient

$$C_{T,N+1} = -2\pi \tag{83}$$

then equation (81) can be written more compactly as follows

$$\sum_{n=1}^{N+1} C_{T,n}.\gamma P_n = 2\pi.V_\infty.\left(XT.\sin\alpha - YT.\cos\alpha\right) \tag{84}$$

All the values of $C_{T,n}$ and the right hand side of the above equation are known and the only unknowns are γP_n for n = 1 to n = N+1 provided that the control point T is chosen to be on the aerofoil's surface such that ψ_T is a constant. Since there are N+1 unknowns therefore we need N+1 equations to be solved simultaneously to calculate the values of the unknowns, γP_n. However there are only N points on the surface of the aerofoil that can be selected to be the control points where equation (84) is applied, thus we need one more equation to complete the system of equations to be solved simultaneously.

The extra equation is obtained from the physical observation that the airflow leaving the upper surface of the aerofoil must have exactly the same velocity as the airflow leaving the lower surface. This means that the velocity at the two tail points (upper and lower tail points) must be the same. It can be shown that the airflow velocity at an aerofoil's surface point is exactly the same as the value of the vortex strength per unit length, γP, at the point. This trailing edge flow condition is known as the Kutta condition and can be represented by the following equation

$$\gamma P_1 + \gamma P_N = 0 \tag{85}$$

For each of the control point, which is chosen to be the aerofoil's surface point P, we can write down an equation based on the general expression of equation (84). The $(N+1)^{th}$ must be derived differently, namely it is based on satisfying the Kutta condition. Now we can define the following for the $(N+1)^{th}$ equation

$$\begin{aligned} &C_{T,1} = 1 \\ &C_{T,n} = 0 \quad for\, n = 2,3,...,N-1 \\ &C_{T,N} = 1 \\ &C_{T,N+1} = 0 \end{aligned} \tag{86}$$

With the above definitions we now have a system of equations consisting of (N+1) equations involving (N+1) unknowns as follows

$$\sum_{n=1}^{N+1} C_{T,n}.\gamma P_n = D_n \tag{87}$$

where

$$\begin{aligned} &D_n = 2\pi.V_\infty.\left(XT_n.\sin\alpha - YT_n.\cos\alpha \right) \quad for\;\, n = 1,2,....,N \\ &D_{N+1} = 0 \end{aligned} \tag{88}$$

The system of equations (87) can be solved simultaneously to calculate the unknown γP_n and since the absolute value of γP_n is the same as the airflow velocity at the point P_n, therefore the distribution of flow velocity along the aerofoil's surface is known. Furthermore, it can be shown that for vortex panel method as described here, the value of pressure coefficient Cp can be obtained from the Bernoulli equation and the final result is

$$Cp = 1 - \left(\frac{\gamma}{V_\infty} \right)^2 \qquad (89)$$

The pressure coefficient at point on the surface of the aerofoil can thus be calculated and plotted as desired. The pressure coefficients can also be integrated to give the resultant force acting on the aerofoil. This resultant force can be resolved into 2 components, one being the lift force in the direction normal to the free stream direction and the other is the drag force acting along the free stream direction. Due to the inviscid flow assumption, it is expected that the drag force or drag coefficient must have a value of zero. In practice the computed drag coefficient will be found to have a non-zero value due to computational error such as round off error etc. The moment acting on the aerofoil can also be obtained from the known pressure distribution. A simple example of the application of the panel method is given in appendix 3.

Even though the panel method is quite good for computing the aerodynamic properties of an aerofoil, it is basically a numerical method and doesn't give analytical insight into the aerodynamic behaviour of an aerofoil. To get such an insight we need an analytical tool, even if it is very much simplified. This tool is known as the Thin Aerofoil Theory, which is the next topic to be discussed.

6. *Thin Aerofoil Theory*

The camber line of an aerofoil is the curve midway between the lower and upper surfaces of the aerofoil. For a symmetric aerofoil the camber line is simply a straight line. If a line is drawn perpendicular to the camber line, then it must intersect both the upper and the lower surfaces of the aerofoil. The distance between the two intersection points is called the thickness of the aerofoil. Obviously this thickness would vary along the chord of the aerofoil. The thickness is normally expressed as a percentage of the chord length. The thickness of an aerofoil is defined as the maximum thickness as described previously. A thin aerofoil is defined as any aerofoil whose (maximum) thickness is a very small percentage of the chord length, such that it is reasonable to model the aerofoil as a curved or flat plate of zero thickness. While this restriction is quite severe, it enables us to get an analytical solution to the problem. Here we sacrifice accuracy to get analytical insight.

6.1 *Flat plate as an aerofoil*

Let us now consider a very thin symmetric aerofoil, which is quite reasonable to be approximated as a flat plate of zero thickness, with a chord length c. The aerofoil is located along the x-axis with its nose being at the origin of the system of coordinates. The aerofoil is immersed in a uniform wind whose velocity vector is at an angle α relative to the horizontal or x-axis. The disturbance to the uniform wind due to the presence of the aerofoil is modelled as the disturbance caused by a vortex sheet located along the camber line of the aerofoil. The vortex strength per unit length distribution of the sheet $\gamma(x)$ is unknown and must be evaluated.

The stream function at any point $\left(x^*, y^*\right)$ within the flow field is given as follows

$$\psi\left(x^*, y^*\right) = V_\infty\left(-x^*.\sin\alpha + y^*.\cos\alpha\right) + \tfrac{1}{2\pi}\int_0^c \gamma(x).\ln r(x).dx \tag{90}$$

$$r(x) = \sqrt{\left(x^* - x\right)^2 + \left(y^* - y\right)^2} \tag{91}$$

The velocity components at any point are given by the following

$$u\left(x^*, y^*\right) = V_\infty.\cos\alpha + \tfrac{1}{2\pi}\int_0^c \gamma(x).\frac{\sin\theta(x)}{r(x)}.dx \tag{92}$$

$$v\left(x^*, y^*\right) = V_\infty.\sin\alpha - \tfrac{1}{2\pi}\int_0^c \gamma(x).\frac{\cos\theta(x)}{r(x)}.dx \tag{93}$$

$$\sin\theta(x) = \frac{y^*}{r(x)} \quad and \quad \cos\theta(x) = \frac{x^* - x}{r(x)} \tag{94}$$

If the problem is specified as a Dirichlet boundary condition problem, then we must find the analytical expression for $\gamma(x)$ such that equation (90) is satisfied at all control or boundary points. The boundary condition to be satisfied is that all the points on the camber line, or part of x-axis from 0 to c, must have the same stream function value because the camber line is a streamline. It is quite easily seen that this problem is impossible to solve analytically.

Now let us recast the problem as a Neuman boundary condition problem. The boundary condition to be satisfied is that the component of velocity normal to the camber line, $v\left(x^*, y^*\right)$, must be zero at all points along the camber line. The equation to be solved is thus equation (93), which can be rewritten as follows

$$V_\infty.\sin\alpha - \tfrac{1}{2\pi}\int_0^c \gamma(x).\frac{\cos\theta(x)}{r(x)}.dx = 0 \tag{95}$$

Since the control point is on the camber line thus y^* is always zero. Therefore $\cos\theta(x)=1$ and $r(x)=x^*-x$.
Equation (95) is then simplified to

$$\tfrac{1}{2\pi}\int_0^c \frac{\gamma(x).dx}{x^*-x}=V_\infty.\sin\alpha \tag{96}$$

It can be seen that this problem looks simpler than the Dirichlet formulation of the same problem. Even so the problem is not quite so simple.
We know that any continuous function can always be approximated by a Fourier series, even if the exact expression for the function is unknown. Therefore, it is reasonable to replace the unknown function $\gamma(x)$ by a Fourier series with unknown coefficients. However, before we can do that it should be realized that the Fourier series is expressed in terms of angles rather than variable such as x. Therefore, it is necessary that we perform a coordinate transformation from x to θ. It is required that when x = 0 then θ= 0 and when x = c we want $\theta=\pi$.

A suitable transformation function can be given as follows

$$\begin{aligned} x &= \tfrac{c}{2}(1-\cos\theta) \\ dx &= \tfrac{c}{2}\sin\theta.d\theta \end{aligned} \tag{97}$$

Equation (95) can now be rewritten as follows

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta).\sin\theta}{\cos\theta-\cos\theta^*}d\theta=V_\infty.\sin\alpha \tag{98}$$

We will not go into the mathematical details of how to solve the above equation. It is sufficient to simply apply the known mathematical results to help find the solution. For example it is known that

$$\begin{aligned} &\int_0^\pi \frac{\cos n\theta.d\theta}{\cos\theta-\cos\theta^*}=\pi\frac{\sin n\theta^*}{\sin\theta^*} \quad for \;\; n=0,1,2,3.... \\ &\int_0^\pi \frac{\sin n\theta.\sin\theta.d\theta}{\cos\theta-\cos\theta^*}=-\pi\cos n\theta^* \quad for \;\; n=0,1,2,3.... \end{aligned} \tag{99}$$

We shall now assume a solution and then substitute the solution into equation (96) and verify that the trial solution indeed satisfies the equation or otherwise.
The trial solution is

$$\gamma(\theta)=2V_\infty.\sin\alpha.\frac{1+\cos\theta}{\sin\theta} \tag{100}$$

The left hand side of equation (96) can now be expanded as follows

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta).\sin\theta}{\cos\theta-\cos\theta^*}d\theta=\frac{1}{2\pi}\int_0^{\pi}\frac{2V_\infty.\sin\alpha.(1+\cos\theta)\sin\theta}{\sin\theta\left(\cos\theta-\cos\theta^*\right)}.d\theta$$

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta).\sin\theta}{\cos\theta-\cos\theta^*}d\theta=\frac{V_\infty.\sin\alpha}{\pi}\int_0^{\pi}\frac{1+\cos\theta}{\cos\theta-\cos\theta^*}.d\theta \tag{101}$$

The first equation in (99) for n = 0 and n = 1 gives the following results

$$\int_0^{\pi}\frac{d\theta}{\cos\theta-\cos\theta^*}=0$$

$$\int_0^{\pi}\frac{\cos\theta.d\theta}{\cos\theta-\cos\theta^*}=\pi \tag{102}$$

Therefore equation (98) can be simplified as follows

$$\frac{V_\infty.\sin\alpha}{\pi}\int_0^{\pi}\frac{1+\cos\theta}{\cos\theta-\cos\theta^*}.d\theta=\frac{V_\infty.\sin\alpha}{\pi}.(0+\pi)=V_\infty.\sin\alpha \tag{103}$$

Thus it has been proven that the trial solution (100) indeed satisfies the governing equation (96).

In the previous section it was stated that the flow solution must also satisfy the Kutta condition at the trailing edge or at $\theta=\pi$ (since x = c). It should be noted that the Kutta condition for an aerofoil with a sharp trailing edge is that the trailing edge must be a rear stagnation point, where $\gamma=V=0$. Substituting the value of $\theta=\pi$ into (100) we get the following result

$$\gamma(\pi)=2V_\infty.\sin\alpha.\frac{0}{0}\quad\text{(indeterminate value).}$$

The above value is indeterminate and we should apply the L'Hospital's rule. The rule states that if the ratio of 2 functions, say f(z)/g(z), becomes indeterminate as z approaches 0, then the value can be calculated by replacing the functions with their derivatives. Thus in the limit of z$\rightarrow$0, the value of $f(0)/g(0)$ is given by $f^{'}(0)/g^{'}(0)$. In our example both f and g approach 0 as $\theta\rightarrow\pi$. Therefore the required value should be calculated as $f^{'}(\pi)/g^{'}(\pi)$.
Now it is noted that

$$\frac{d(1+\cos\theta)}{d\theta}=-\sin\theta\quad\text{and}\quad\frac{d\sin\theta}{d\theta}=\cos\theta$$

Therefore

$$\gamma(\pi) = 2V_\infty.\sin\alpha.\frac{-\sin\pi}{\cos\pi} = 0$$

It can be seen that the solution also satisfies the Kutta condition.

We can also obtain the solution as a function of the Cartesian variable, x, as follows. From equation (97) we can get the following

$$1+\cos\theta = 2(1-x/c)$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = +2\sqrt{x(1-x)/c^2}$$

Therefore

$$\gamma(x) = 2V_\infty.\sin\alpha.\frac{c-x}{\sqrt{x(c-x)}} = 2V_\infty.\sin\alpha.\sqrt{\frac{c-x}{x}} \tag{104}$$

At the trailing edge x = c, therefore $\gamma(c) = 0$, thus satisfying the Kutta condition.

The values of $\gamma^* = \dfrac{\gamma}{V_\infty.2\sin\alpha}$ can be calculated as a function of x/c and the results are tabulated below

x/c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
γ^*	∞	3	2	1.528	1.225	1.000	0.816	0.655	0.500	0.333	0.00

The lift, L, acting on the flat plate can be calculated using the following Kutta-Joukowski lift equation

$$L = \rho V_\infty \Gamma \tag{105}$$

The circulation around the aerofoil is equal to the total strength of the distributed vortex, which can be obtained by integrating or summing up all the values of $\gamma(x)$.

$$\Gamma = \int_0^c \gamma(x).dx = \frac{c}{2}\int_0^\pi \gamma(\theta).\sin\theta.d\theta$$

(106)

$$\Gamma = cV_\infty \sin\alpha \int_0^\pi (1+\cos\theta)d\theta = \pi c V_\infty \sin\alpha$$

The lift coefficient is defined as follows

$$C_l = \frac{L}{\frac{1}{2}\rho V_\infty^2 .c} = 2\pi \sin\alpha \quad (107)$$

For small values of α (in radian) the following approximation is quite accurate

$$\sin\alpha \doteq \alpha \quad (108)$$

Therefore the lift coefficient can be rewritten as follows

$$C_l = 2\pi.\alpha \quad (109)$$

The force dF acting on an aerofoil elemental length dx is given by the Kutta-Joukowski lift equation as follows

$$dF = \rho V_\infty \gamma(x).dx \quad (110)$$

The pitching moment about the leading edge due to the force dF is

$$M_{LE} = -\int_0^c x.dF = -\int_0^c \rho V_\infty \gamma(x).x.dx$$

$$M_{LE} = -\int_0^\pi \rho V_\infty .2V_\infty \alpha \frac{1+\cos\theta}{\sin\theta}.\frac{c}{2}(1-\cos\theta).\frac{c}{2}\sin\theta d\theta$$

$$M_{LE} = -\frac{c^2}{2}\rho V_\infty^2 \alpha \frac{\pi}{2} = -\frac{\pi}{4}\alpha\rho V_\infty^2 c^2 \quad (111)$$

The pitching moment about any other point can be obtained quite easily by remembering the definition of moment, which is simply force multiplied by the arm length. The pressure distribution along the aerofoil surface is such that it can be replaced by a resultant moment and a resultant force (lift force). There is a particular location or value of x, where the resultant moment of the pressure distribution is zero. This zero resultant moment point is known as the centre of pressure and its location can be calculated as follows.

Let us assume that the centre of pressure is located at x_{cp}. At this point the resultant moment is zero and the resultant force is given by equation (105). Now imagine that an equal but opposite force is located at the same point. To cancel this force we must add another force, which is equal to the resultant force but located elsewhere, say at the leading edge. Adding two equal but opposite forces doesn't change the resultant force acting on the aerofoil, but those forces which is called a couple is equivalent to a moment. The magnitude of this moment of the couple is simply x_{cp} multiplied by the lift force. However, to get the moment acting at the centre of pressure to remain zero we must add a moment which is equal but in opposite direction of the moment due to the couple. Now we have the lift force at the centre of pressure, a couple of forces the magnitude of which is the same as the lift force and a moment, which has a magnitude of $x_{cp}.L$ in the direction that the lift force would rotate the aerofoil about the leading edge as the axis of rotation. This of course is in the counter clockwise direction. A moment is defined as being positive if it tends to rotate the aerofoil nose up. Therefore the moment acting at the leading edge is negative. A lift force acting upwards is defined to be positive and thus the value of x_{cp} can be calculated as follows

$$x_{cp} = -\frac{M_{LE}}{L} = -\frac{-\frac{\pi}{4}.\alpha\rho V_\infty^2.c^2}{\pi\rho V_\infty^2 c.\alpha} = \frac{c}{4} \tag{112}$$

The coefficient of moment about the leading edge is defined as

$$C_{m,LE} = \frac{M_{LE}}{\frac{1}{2}\rho V_\infty^2 c^2} = \frac{-\frac{1}{4}\pi\alpha\rho V_\infty^2 c^2}{\frac{1}{2}\rho V_\infty^2 c^2} = -\frac{1}{2}\pi\alpha \tag{113}$$

It should be noted that x_{cp} can also be obtained from the following equation

$$\frac{x_{cp}}{c} = -\frac{C_{m,LE}}{C_l} = -\frac{-\frac{1}{2}\pi\alpha}{2\pi\alpha} = \frac{1}{4} \tag{114}$$

6.2 *Cambered thin aerofoil*

In the previous section it has been shown how a simple analysis based on a very crude model with severe restrictions can still be very useful in giving insight into the aerodynamic properties of aerofoils. Whilst the flat plate model for an aerofoil is not capable of giving detailed knowledge of velocity or pressure distribution around the aerofoil surface with any accuracy, it is capable of predicting reasonably accurately the value of the slope of the lift versus angle of attack curve, namely 2π . This value is quite close to the value obtained from wind tunnel measurement or much more sophisticated numerical modelling, which gives slightly lower value of the slope but differing by a factor of not more than 10 percent or so.
The method is also capable of predicting the location of the centre of pressure, which for this simplified flow model is the same as the aerodynamic centre. For the case of aerofoils in actual subsonic flows, the aerodynamic centre is located near the quarter chord point as predicted by the flat plate model. However, the centre of pressure moves as a function of the angle of attack, α .

In this section we shall relax the constraints slightly by allowing the aerofoil to have curvature or camber. The thickness of the aerofoil must still be very small, and the maximum camber must also be very small such that the aerofoil is still very much like a flat plate. However, because the aerofoil is allowed to have camber the boundary condition that must be satisfied is somewhat different from the flat plate model. At a point on the camber line, which represents the aerofoil's surface, the normal to the camber line is not exactly in the y-direction (as in the case of the flat plate model) but along a line that is slightly inclined to the y-axis. Let this angle be η and is the same as the inclination of the tangent (or slope) to the camber line at the control point. This implies that

$$\tan\eta = \frac{dy}{dx} \qquad \text{or} \qquad \eta = \tan^{-1}\left(\frac{dy}{dx}\right) \tag{115}$$

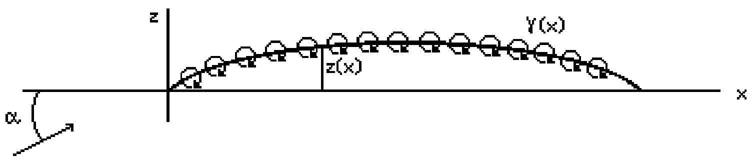
It is further assumed that the angle is sufficiently small such that the tangent of the angle is equal to the angle itself in unit of radian, i.e. $\eta \simeq \tan\eta = \frac{dy}{dx}$.

The equation to be satisfied is now slightly different from equation (95) as follows

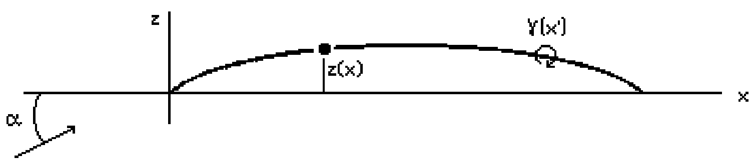
$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x).dx}{x^* - x} = V_\infty.\sin(\alpha - \eta) \tag{116}$$

Since $(\alpha - \eta)$ is quite small, therefore the above equation can be simplified as follows

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x).dx}{x^* - x} = V_\infty.(\alpha - \eta) \tag{117}$$



From the web site http://www.desktopaero.com/appliedaero/airfoils1/tatderivation.html



From the web site http://www.desktopaero.com/appliedaero/airfoils1/tatderivation.html

The solution of equation (116) has 2 components. The first one is the same solution as for the flat plate situation or equation (98). The second component is to account for the aerofoil's camber. The trial solution is thus given as follows

$$\gamma(\theta) = 2V_\infty\left[A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right] \tag{118}$$

Substituting (118) into (117) we get

$$\frac{1}{\pi}\int_0^\pi \left(A_0\frac{1+\cos\theta}{\cos\theta - \cos\theta^*} + \sum_{n=1}^{\infty} A_n \frac{\sin n\theta.\sin\theta}{\cos\theta - \cos\theta^*}\right).d\theta = \alpha - \eta(\theta) \tag{119}$$

From equation (99) it is known that

$$\frac{1}{\pi}\int_0^\pi \frac{1+\cos\theta}{\cos\theta - \cos\theta^*}.d\theta = 1 \tag{120}$$

The second part of equation (99) gives the following results

$$\int_0^\pi \frac{\sin n\theta.\sin\theta}{\cos\theta-\cos\theta^*}.d\theta = -\pi.\cos n\theta^* \quad \text{for n = 1, 2,…..} \tag{121}$$

Substituting (120) and (121) into (119) we get the following

$$A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta^* = \alpha - \eta\left(\theta^*\right) \tag{122}$$

Therefore our trial solution (118) satisfies the governing boundary condition (117) provided the Fourier series coefficients satisfy the requirement of equation (122).

We know from basic calculus that

$$\cos\left(n+m\right)\theta^* = \cos n\theta^* \cos m\theta^* - \sin n\theta^* \sin m\theta^* \tag{123}$$

$$\cos\left(n-m\right)\theta^* = \cos n\theta^* \cos m\theta^* + \sin n\theta^* \sin m\theta^* \tag{124}$$

Therefore $$\tfrac{1}{2}\left(\cos\left(n+m\right)\theta^* + \cos\left(n-m\right)\theta^*\right) = \cos n\theta^* \cos m\theta^* \tag{125}$$

The results above can be used to get the following results

$$\begin{aligned} \int_0^\pi \cos n\theta^* \cos m\theta^* d\theta^* &= \tfrac{1}{2}\int_0^\pi \left(\cos\left(n+m\right)\theta^* + \cos\left(n-m\right)\theta^*\right) d\theta^* \\ &= \frac{\pi}{2} \quad \text{if } n=m \text{ or } \left(n+m\right)=0 \\ &= 0 \quad \text{if } n \neq m \end{aligned} \tag{126}$$

Integrating equation (122) from 0 to π, we can get the following

$$\int_0^\pi \left(A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta^* \right) d\theta^* = \int_0^\pi \left(\alpha - \eta\left(\theta^*\right)\right).d\theta^*$$

Since $\int_0^\pi \cos n\theta^* d\theta^* = 0$ for n = 1, 2, 3,…, therefore

$$A_0 = \frac{1}{\pi}\int_0^\pi \left(\alpha - \eta\left(\theta^*\right)\right).d\theta^* = \alpha - \frac{1}{\pi}\int_0^\pi \eta\left(\theta^*\right).d\theta^* \tag{127}$$

Multiplying equation (122) by $\cos m\theta^*$ and integrating from 0 to π we get

$$A_0\int_0^\pi \cos m\theta^* d\theta^* - \sum_{n=1}^{\infty} A_n \int_0^\pi \cos n\theta^* \cos m\theta^* d\theta^* = \int_0^\pi \left(\alpha - \eta\left(\theta^*\right)\right)\cos m\theta^*.d\theta^*$$

The integrals on the left hand side of the equation above are all zero except if $n = m$, in which case it has the value of $\frac{\pi}{2}$. The first term on the right hand side of the equation is always zero hence the equation can be simplified as follows

$$A_m = \frac{2}{\pi}\int_0^{\pi} \eta\left(\theta^*\right).\cos m\theta^*.d\theta^* \quad \text{for } \text{m} = 1, 2, 3, \ldots.. \tag{128}$$

Since in this last result m is just a dummy index, thus we can change it to n or k or any other symbol if we so wish.
The total circulation around the aerofoil is given by equation (106), which for cambered aerofoil can be written as follows

$$\Gamma = V_\infty.c\left[A_0\int_0^{\pi}\left(1+\cos\theta\right).d\theta + \sum_{n=1}^{\infty} A_n\int_0^{\pi}\sin n\theta.\sin\theta.d\theta\right] \tag{129}$$

From basic mathematics we know that

$$\sin n\theta.\sin\theta = \tfrac{1}{2}\left(\cos\left(n-1\right)\theta - \cos\left(n+1\right)\theta\right) \tag{130}$$

$$\sin n\theta.\sin 2\theta = \tfrac{1}{2}\left(\cos\left(n-2\right)\theta - \cos\left(n+2\right)\theta\right) \tag{131}$$

Therefore

$$\int_0^{\pi}\sin n\theta.\sin\theta.d\theta = \frac{\pi}{2} \quad \text{for n = 1} \tag{132}$$

$$= 0 \quad \text{if } n \neq 1 \tag{133}$$

$$\int_0^{\pi}\sin n\theta.\sin 2\theta.d\theta = \frac{\pi}{2} \quad \text{for n = 2} \tag{134}$$

$$= 0 \text{ if } n \neq 2 \tag{135}$$

Equation (129) can now be simplified as follows

$$\Gamma = V_\infty.c.\left(A_0\pi + A_1.\frac{\pi}{2}\right) = V_\infty.c.\frac{\pi}{2}\left(2A_0 + A_1\right) \tag{136}$$

The lift acting on the aerofoil is given by the Kutta-Joukowski lift equation (120), hence the lift coefficient can be calculated as follows

$$C_l = \frac{\frac{1}{2}\rho V_\infty^2.c.\pi\left(2A_0 + A_1\right)}{\frac{1}{2}\rho V_\infty^2.c} = 2\pi\left(A_0 + \frac{1}{2}A_1\right) \tag{137}$$

The lift curve slope can be calculated knowing that A_0 is given by equation (127)

$$C_{l,\alpha} = \frac{\partial C_l}{\partial\alpha} = 2\pi \tag{138}$$

The pitching moment about the leading edge can also be evaluated as follows

$$M_{LE} = -\int_0^c x.dF = -\int_0^c \rho V_\infty \gamma(x).x.dx$$

where $\gamma(x) = \gamma(\theta)$ is given by equation (96) and x and dx are given by equation (112), hence the above equation can be simplified further as follows

$$M_{LE} = -\tfrac{1}{2}\rho V_\infty^2 c^2 \int_0^\pi \left[A_0\left(1-\cos^2\theta\right) + \sum_{n=1}^\infty A_n \sin n\theta.\sin\theta\left(1-\cos\theta\right) \right] d\theta$$

$$M_{LE} = -\tfrac{1}{2}\rho V_\infty^2 c^2 \int_0^\pi \left[\tfrac{1}{2}A_0\left(1-\cos 2\theta\right) + \sum_{n=1}^\infty A_n\left(\sin n\theta.\sin\theta - \tfrac{1}{2}\sin n\theta.\sin 2\theta\right) \right] d\theta$$

Substituting equations (149), (150), (151) and (152) into the above, we finally get

$$M_{LE} = -\tfrac{1}{2}\rho V_\infty^2 c^2 \left[A_0\frac{\pi}{2} + A_1\frac{\pi}{2} - A_2\frac{\pi}{4} \right] = -\tfrac{1}{2}\rho V_\infty^2 c^2 \frac{\pi}{2}\left[A_0 + A_1 - \tfrac{1}{2}A_2\right]$$

The leading edge moment coefficient is then

$$C_{m,LE} = \frac{M_{LE}}{\tfrac{1}{2}\rho V_\infty^2 c^2} = -\frac{\pi}{2}\left[A_0 + A_1 - \tfrac{1}{2}A_2\right] \qquad (139)$$

The location of the centre of pressure is given by

$$\frac{x_{cp}}{c} = -\frac{C_{m,LE}}{C_l} = \frac{1}{4}\frac{A_0 + A_1 - \tfrac{1}{2}A_2}{A_0 + \tfrac{1}{2}A_1} \qquad (140)$$

The moment coefficient at the quarter chord point is

$$C_{m,c/4} = C_{m,LE} + \tfrac{1}{4}C_l = -\tfrac{\pi}{2}\left[A_0 + A_1 - \tfrac{1}{2}A_2\right] + \tfrac{\pi}{2}\left(A_0 + \tfrac{1}{2}A_1\right) = \tfrac{\pi}{4}\left(A_2 - A_1\right) \qquad (141)$$

The equations for the camber lines for NACA aerofoils can be obtained from the book by I.H.Abbott and A.E.von Doenhoff : *Theory of Wing Sections* or it can be obtained from the following web address
http://www.desktopaero.com/appliedaero/appliedaero.html

As an example of the application of TAT for a cambered aerofoil we will consider a simple aerofoil whose camber is given by $y(x) = 4h.\frac{x}{c}.\left(1-\frac{x}{c}\right)$ where the maximum camber h is a small positive number and c is the chord

hence $$\frac{dy}{dx} = 4\frac{h}{c}\left(1-2\frac{x}{c}\right) = 4\frac{h}{c}\left(1-\left(1-\cos\theta\right)\right) = 4\frac{h}{c}\cos\theta \qquad (142)$$

The Fourier coefficients for this aerofoil can be calculated as follows

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi}\frac{4h}{c}\cos\theta d\theta = \alpha$$

$$A_1 = \frac{2}{\pi}\frac{4h}{c}\int_0^{\pi}\cos\theta.\cos\theta.d\theta = \frac{8h}{\pi c}.\frac{\pi}{2} = \frac{4h}{c}$$

$$A_2 = \frac{2}{\pi}\frac{4h}{c}\int_0^{\pi}\cos 2\theta.\cos\theta.d\theta = \frac{8h}{\pi c}.0 = 0$$

Note that equation (141) has been used to evaluate the integrals for A_1 and A_2.
The value of the lift coefficient is

$$C_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right) = 2\pi\left(\alpha + \frac{2h}{c}\right) \tag{143}$$

The leading edge and quarter chord moment coefficients, and the centre of pressure location are

$$C_{m,LE} = -\frac{\pi}{2}\left(\alpha + \frac{4h}{c}\right) \tag{144}$$

$$C_{m,c/4} = -\pi\frac{h}{c} \tag{145}$$

$$\frac{x_{cp}}{c} = \frac{1}{4}\frac{\alpha + 4h/c}{\alpha + 2h/c} = \frac{1}{4}\left(1 + 2\frac{h/c}{\alpha + 2h/c}\right) \tag{146}$$

Let us now study a similar aerofoil, except that the maximum camber is now located at 0.25c, rather than at the mid point 0.5c.
The equation for the camber of the aerofoil is now given by the following

For $0 \le x \le \tfrac{1}{4}c$: $y(x) = h.\frac{8x(c-2x)}{c^2}$ and $\frac{dy}{dx} = \frac{8h}{c^2}(c-4x)$ (147)

Therefore $\frac{dy}{dx} = \frac{8h}{c}\left(1 - 4\left(\tfrac{1}{2}(1-\cos\theta)\right)\right) = \frac{8h}{c}(2\cos\theta - 1)$ (148)

For $\tfrac{1}{4}c \le x \le c$: $y(x) = 8h.\frac{(2x+c)(c-x)}{9c^2}$ and $\frac{dy}{dx} = \frac{8h}{9c^2}(c-4x)$ (149)

Therefore $\frac{dy}{dx} = \frac{8h}{9c}\left(1 - 4\left(\tfrac{1}{2}(1-\cos\theta)\right)\right) = \frac{8h}{9c}(2\cos\theta - 1)$ (150)

It should be noted that x/c =1/4 is equivalent to $\cos\theta = 0.5$ or $\theta = \pi/3$

The Fourier coefficients can now be calculated as follows

$$A_0 = \alpha - \frac{8h}{\pi c}\left[\int_0^{\pi/3}\left(2\cos\theta - 1\right).d\theta + \frac{1}{9}\int_{\pi/3}^{\pi}\left(2\cos\theta - 1\right).d\theta\right]$$

$$A_0 = \alpha - \frac{8h}{\pi c}\left[\left(2\sin\theta - \theta\right)\Big|_0^{\pi/3} + \frac{1}{9}\left(2\sin\theta - \theta\right)\Big|_{\pi/3}^{\pi}\right]$$

$$A_0 = \alpha - \frac{8h}{\pi c}\left[0.6849 - \frac{1}{9}x\,3.8265\right] = \alpha - 0.6613\frac{h}{c}$$

$$A_1 = \frac{16h}{\pi c}\left[\int_0^{\pi/3}\left(2\cos^2\theta - \cos\theta\right).d\theta + \frac{1}{9}\int_{\pi/3}^{\pi}\left(2\cos^2\theta - \cos\theta\right).d\theta\right]$$

$$A_1 = \frac{16h}{\pi c}\left[\left(\theta + \frac{1}{2}\sin 2\theta - \sin\theta\right)\Bigg|_0^{\pi/3} + \frac{1}{9}\left(\theta + \frac{1}{2}\sin 2\theta - \sin\theta\right)\Bigg|_{\pi/3}^{\pi}\right]$$

$$A_1 = \frac{16h}{\pi c}\left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} - \sin\frac{\pi}{3} + \frac{1}{9}\left(\frac{2\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \sin\frac{\pi}{3}\right)\right]$$

$$A_1 = \frac{16h}{\pi c}\left[0.6142 + \frac{2.5274}{9}\right] = 4.5583\frac{h}{c}$$

$$A_2 = \frac{16h}{\pi c}\left[\int_0^{\pi/3}\left(2\cos 2\theta.\cos\theta - \cos 2\theta\right).d\theta + \frac{1}{9}\int_{\pi/3}^{\pi}\left(2\cos 2\theta.\cos\theta - \cos 2\theta\right).d\theta\right]$$

$$A_2 = \frac{16h}{\pi c}\left[\left(\frac{1}{3}\sin 3\theta + \sin\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\pi/3} + \frac{1}{9}\left(\frac{1}{3}\sin 3\theta + \sin\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_{\pi/3}^{\pi}\right]$$

$$A_2 = \frac{16h}{\pi c}\left[\sin\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{1}{9}\left(-\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)\right]$$

$$A_2 = \frac{16h}{\pi c}\left[0.4330 - \frac{1}{9}x\,0.4330\right] = 1.9602\frac{h}{c}$$

The lift coefficient can now be calculated as follows

$$C_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right) = 2\pi\left(\alpha - \frac{h}{c}\left(0.6613 - 2.2792\right)\right) = 2\pi\left(\alpha + 1.6179\frac{h}{c}\right) \qquad (151)$$

The pitching moment coefficients and the centre of pressure location are

$$C_{m,LE} = -\frac{\pi}{2}\left[\alpha - 0.6613\frac{h}{c} + \left(4.5583 - 0.9801\right)\frac{h}{c}\right] = -\frac{\pi}{2}\left[\alpha + 2.9169\frac{h}{c}\right] \qquad (152)$$

$$C_{m,c/4} = -0.6495\pi \frac{h}{c} \quad (153)$$

$$\frac{x_{cp}}{c} = -\frac{C_{m,LE}}{C_l} = \frac{1}{4}\frac{\alpha + 2.9169h/c}{\alpha + 1.6179h/c} = \frac{1}{4}\left(1 + 1.299\frac{h/c}{\alpha + 1.6179h/c}\right) \quad (154)$$

Now let us consider another situation where the maximum camber location is pushed back even further. For the cases where the maximum camber is located at x/c = ¾, the results are as follows

For $0 \le x \le \frac{3}{4}c$: $y(x) = \frac{8h}{9}.\frac{x(3c-2x)}{c^2}$ and $\frac{dy}{dx} = \frac{8h}{9c}\frac{(3c-4x)}{c}$ (156)

Therefore $\frac{dy}{dx} = \frac{8h}{9c}\left(3 - 4\left(\frac{1}{2}(1-\cos\theta)\right)\right) = \frac{8h}{9c}(2\cos\theta + 1)$ (157)

For $\frac{3}{4}c \le x \le c$: $y(x) = \frac{8h}{c}.\frac{(2x-c)(c-x)}{c}$ and $\frac{dy}{dx} = \frac{8h}{c}\frac{(3c-4x)}{c}$ (158)

Therefore $\frac{dy}{dx} = \frac{8h}{c}\left(1 - 4\left(\frac{1}{2}(1-\cos\theta)\right)\right) = \frac{8h}{c}(2\cos\theta + 1)$ (159)

It should be noted that x/c =3/4 is equivalent to $\cos\theta = -0.5$ or $\theta = 2\pi/3$
The Fourier coefficients can now be calculated as follows

$$A_0 = \alpha - \frac{8h}{9\pi c}\left[\int_0^{2\pi/3}(2\cos\theta + 1).d\theta + 9.\int_{2\pi/3}^{\pi}(2\cos\theta + 1).d\theta\right]$$

$$A_0 = \alpha - \frac{8h}{9\pi c}\left[(2\sin\theta + \theta)\Big|_0^{2\pi/3} + 9.(2\sin\theta + \theta)\Big|_{2\pi/3}^{\pi}\right]$$

$$A_0 = \alpha - \frac{8h}{9\pi c}\left[3.8264 - 9x0.6849\right] = \alpha + 2.078\frac{h}{c}$$

$$A_1 = \frac{16h}{9\pi c}\left[\int_0^{2\pi/3}\left(2\cos^2\theta + \cos\theta\right).d\theta + 9.\int_{2\pi/3}^{\pi}\left(2\cos^2\theta + \cos\theta\right).d\theta\right]$$

$$A_1 = \frac{16h}{9\pi c}\left[\left(\theta + \frac{1}{2}\sin 2\theta + \sin\theta\right)\Big|_0^{2\pi/3} + 9.\left(\theta + \frac{1}{2}\sin 2\theta + \sin\theta\right)\Big|_{2\pi/3}^{\pi}\right]$$

$$A_1 = \frac{16h}{9\pi c}(3.3934 + 9x0.9023) = 6.5156\frac{h}{c}$$

$$A_2 = \frac{16h}{9\pi c}\left[\int_0^{2\pi/3}\left(2\cos 2\theta.\cos\theta + \cos 2\theta\right).d\theta + 9.\int_{2\pi/3}^{\pi}\left(2\cos 2\theta.\cos\theta + \cos 2\theta\right).d\theta\right]$$

$$A_2 = \frac{16h}{9\pi c}\left[\left(\frac{1}{3}\sin 3\theta + \sin\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{2\pi/3} + 9.\left(\frac{1}{3}\sin 3\theta + \sin\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{2\pi/3}^{\pi}\right]$$

$$A_2 = \frac{16h}{9\pi c}\left[0 + \sin\frac{2\pi}{3} + \frac{1}{2}\sin\frac{4\pi}{3} + 9.\left(0 - \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3}\right)\right]$$

$$A_2 = \frac{16h}{9\pi c}\left[\frac{1}{2}\sin\frac{\pi}{3} - 9x\frac{1}{2}\sin\frac{\pi}{3}\right] = -1.9603\frac{h}{c}$$

The lift coefficient can now be calculated as follows

$$C_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right) = 2\pi\left(\alpha + \frac{h}{c}(2.078 + 3.2578)\right) = 2\pi\left(\alpha + 5.3358\frac{h}{c}\right) \tag{160}$$

The leading edge pitching moment and centre of pressure location are

$$C_{m,LE} = -\frac{\pi}{2}\left[\alpha + 2.078\frac{h}{c} + (6.5156 + 0.9801)\frac{h}{c}\right] = -\frac{\pi}{2}\left[\alpha + 9.5738\frac{h}{c}\right] \tag{161}$$

$$C_{m,c/4} = \tfrac{\pi}{4}(-1.9603 - 6.5156)(h/c) = -2.119\pi.(h/c) \tag{162}$$

$$\frac{x_{cp}}{c} = -\frac{C_{m,LE}}{C_l} = \frac{1}{4}.\frac{\alpha + 9.5738(h/c)}{\alpha + 5.3358(h/c)} = \frac{1}{4}\left(1 + 4.238\frac{(h/c)}{\alpha + 5.3358(h/c)}\right) \tag{163}$$

The various results can be tabulated to see the effects of shifting the location of the maximum camber point as follows.

Max. camber location	0.25 c	0.50 c	0.75 c
$C_l / 2\pi$	$\alpha + 1.6179(h/c)$	$\alpha + 2.0(h/c)$	$\alpha + 5.3358(h/c)$
$-(2/\pi)C_{m.LE}$	$\alpha + 2.9169(h/c)$	$\alpha + 4.0(h/c)$	$\alpha + 9.5738(h/c)$
$\frac{x_{cp}}{c/4} - 1$	$1.299\frac{(h/c)}{\alpha + 1.6179(h/c)}$	$2\frac{(h/c)}{\alpha + 2.0(h/c)}$	$4.238\frac{(h/c)}{\alpha + 5.3358(h/c)}$
$-C_{m,c/4}/\pi$	$0.6495(h/c)$	(h/c)	$2.119(h/c)$

From the table it can be seen that as the location of the maximum camber is moved backward towards the trailing edge, for a given value of maximum camber, h/c, and angle of attack, α, then the lift coefficient increases and the leading edge pitching moment also increases or becomes more negative. The quarter chord moment also becomes more negative, but the location of the centre of pressure moves further

backward by an amount, which depends on both h/c and α as well as on the location of the maximum camber.
If the maximum camber location is fixed, then for a given angle of attack the lift coefficient and the moment coefficients all increase with increasing value of the maximum camber. However the location of the centre of pressure behaves in a more complex manner and depends on the magnitude of the angle of attack as well.

If the maximum camber, h/c, and its location are kept constant, then the lift coefficient and the leading edge moment coefficient both increase linearly with increasing angle of attack. The quarter chord moment remains constant and is independent of the angle of attack. However, the location of the centre of pressure again behaves in a complex manner depending on the relative values of the maximum camber, its location and the angle of attack.

If the angle of attack is zero, the above table can be simplified as follows

Max. camber location	0.25 c	0.50 c	0.75 c
$\frac{C_l}{2\pi}$	$1.6179\frac{h}{c}$	$2.0\frac{h}{c}$	$5.3358\frac{h}{c}$
$-\frac{2}{\pi}C_{m.LE}$	$2.9169\frac{h}{c}$	$4.0\frac{h}{c}$	$9.5738\frac{h}{c}$
$\frac{x_{cp}}{c}$	0.4508	0.500	0.4485

As can be seen from the above table, an aerofoil seems to have a positive value of lift coefficient even if the angle of attack is zero. This implies that at a particular negative value of angle of attack, the lift acting on an aerofoil is zero. The angle of attack for a cambered aerofoil when the lift produced is zero is called the zero lift angle of attack and is denoted by the symbol of α_0. The lift coefficient for a cambered aerofoil is usually given as

$$C_l = 2\pi(\alpha - \alpha_0) \tag{164}$$

From the above table it can be seen that the magnitude of zero lift angle of attack for a given camber, h/c, increases monotonically as the location of that maximum camber is moved further and further backward towards the trailing edge. If the location of the maximum camber is fixed, both the lift and moment coefficients increase linearly with increasing camber, h/c. Furthermore, the constant of proportionality increases rapidly with increasing value of the location of the maximum camber (or the further rearwards the maximum camber point is located on the aerofoil). The magnitude of α_0 is obviously determined by the curvature of the aerofoil, which is dependant on both the magnitude and location of the maximum camber h/c.

Max. camber location	0.25c	0.50c	0.75c
α_0 (for h/c = 0.01)	-0.93^0	-1.15^0	-3.06^0
α_0 (for h/c = 0.02)	-1.85^0	-2.29^0	-6.11^0
α_0 (for h/c = 0.03)	-2.78^0	-3.44^0	-9.17^0

A rather interesting point to note is the complex behaviour of the location of the centre of pressure. From previous result we know that the centre of pressure for a symmetric aerofoil (or flat plate) is located at the quarter chord point. However, from the above table it can be seen that for zero angle of attack, the centre of pressure location for a cambered aerofoil is actually near the mid point of the aerofoil rather than the quarter chord point. Furthermore, that location is independent of the value of the camber, but is slightly dependent on the location of the maximum camber point. For an aerofoil with its maximum camber located at the chord's mid point, the centre of pressure is also located at the mid point. However, if the location of the maximum camber is moved either forward or backward of the mid point, the centre of pressure always moves slightly forward of the mid point.

The centre of pressure is a concept in Mechanics, which is useful in presenting data about distributed stresses or pressure in the most efficient way. All the relevant information about the distributed pressure can be presented in terms of only 2 quantities, namely the resultant force and the location of the centre of pressure. The centre of pressure is defined as the point about which the moment of the distributed pressure is zero. It can also be thought of as being the point at which the resultant force acts. The moment at any other point can be calculated simply as the product of the resultant force multiplied by the distance of the point from the centre of pressure. The concept of the centre of pressure is, however, sometimes rather useless in the field of aerodynamics. In mechanics the resultant force is normally quite large in magnitude. On the other hand, in aerodynamics the resultant force is often quite small and sometimes even zero. When the lift, or total force, is zero then the definition of the centre of pressure is quite meaningless. When lift is zero the resultant of the distributed pressure is a pure couple, which of course has a constant moment value at any point along the chord of the aerofoil. There is no point where the moment acting on the aerofoil is zero, so there is no centre of pressure. To get out of this difficulty, in aerodynamics we define the aerodynamic centre as the point about which the moment acting on the aerofoil is constant, independent of angle of attack or the lift produce.

The information about the distributed pressure can then be represented by 3 quantities, namely the resultant force, the moment about the aerodynamic centre and the location of the aerodynamic centre itself. The result of TAT analysis suggests that the aerodynamic centre is located at the quarter chord point, since as has been pointed out earlier the moment at the quarter chord is independent of angle of attack. Experimental results suggest that real aerofoils in real flows do have their aerodynamic centres located very near the quarter chord point, at least for subsonic flows. (Aerofoils in supersonic flows have their aerodynamic centres located closer to the midpoint of the chord).

Because of this rather fortunate situation, it is not necessary to present the data of the location of the aerodynamic centre. Thus it can be seen that the aerodynamic centre is a very useful concept in aerodynamics, similar to the usefulness of the concept of the centre of pressure in Mechanics. It is for that reason that we will be discussing more about the moment about the aerodynamic centre (or the quarter chord point) rather than the moment about the centre of pressure.

6.3 Flapped Aerofoil and Control Surfaces.

Let us now apply our knowledge to study the behaviour of control surfaces such as the rudder, elevator or aileron. The control surfaces are normally symmetric aerofoils, and here we will impose the constraint that they are very thin so that the results of TAT are applicable for them.
The elevator is the control surface to create pitching moment, and is attached to the horizontal tail plane via a hinge line. Let the horizontal tail plane, including the elevator, be represented by a portion of the x-axis, with its nose located at origin of the coordinates system and its chord length is c. Let the hinge line of the elevator be located at xh = (1-F).c, where F is the length of the elevator as a fraction of the total chord length c and normally has a value of around 0.3 or there about. The elevator may be deflected up or down. To create a nose up or positive pitching moment the elevator must be deflected upwards, and downwards deflection will create a negative pitching moment. A horizontal tail plane with its elevator deflected is modelled very simply as a horizontal straight line from the nose at the origin to the hinge point H at xh and another straight line representing the elevator from the hinge line to the trailing edge. This elevator is at a negative deflection angle of η when deflected downward. Let us also assume that the airflow is moving uniformly at an angle of α relative to the horizontal line.
Transformation of coordinate from Cartesian to polar coordinate gives us the following results. The nose is at $\theta = 0$ and the trailing edge is at $\theta = \pi$, while the hinge line is at θh, where $xh = (1-F)c = (1-\cos\theta h).c/2$ or $\theta h = \cos^{-1}(2F-1)$.
The Fourier series coefficients of the TAT solution for this case can be calculated as follows

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \eta.d\theta \simeq \alpha - \frac{1}{\pi}\int_0^{\pi}\left(\frac{dy}{dx}\right).d\theta \tag{165}$$

Now it should be remembered that for $0 \le \theta \le \theta h$ we have $\eta = 0$ since the line representing the main tail plane is horizontal, whereas for $\theta h \le \theta \le \pi$, i.e. from the hinge line to the trailing edge, we have a constant negative value of η.
This means that the coefficient can be calculated as follows

$$A_0 = \alpha - \frac{1}{\pi}\left[\eta.\int_{\theta h}^{\pi} d\theta\right] = \alpha - \left(1 - \frac{\theta h}{\pi}\right)\eta \tag{166}$$

The other coefficients, for n = 1, 2, 3, …, are given as follows

$$A_n = \frac{2}{\pi}\eta\left[\int_{\theta h}^{\pi} \cos n\theta.d\theta\right] = -\frac{2\sin n.\theta h}{n.\pi}.\eta \tag{167}$$

$$A_1 = -\tfrac{2}{\pi}\sin(\theta h).\eta \tag{168}$$

$$A_2 = -\tfrac{1}{\pi}\sin(2\theta h).\eta = A_1.\cos\theta h \tag{169}$$

Let us now define the following functions of θh

$$k_1(\theta h) = 1 - \frac{\theta h}{\pi} \tag{170}$$

$$k_2(\theta h) = \tfrac{1}{4}(2\sin\theta h - \sin 2\theta h) \tag{171}$$

$$k_3(\theta h) = 1 - \frac{\theta h}{\pi} + \frac{\sin\theta h}{\pi} = k_1(\theta h) + \frac{\sin\theta h}{\pi} \tag{172}$$

The lift and quarter chord moment coefficients of the tail plane with the elevator deflected are then given by the following equations.

$$C_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right) = 2\pi\left(\alpha - k_3(\theta h).\eta\right) \tag{173}$$

$$C_{m,c/4} = -\tfrac{1}{4}(A_1 - A_2) = k_2(\theta h)\eta \tag{174}$$

The pressure that acts on the elevator would produce a turning moment about the hinge axis. If the pilot wishes to change the elevator deflection angle setting, she or he has to exert a force on the control stick to overcome the elevator's hinge moment. It is useful, therefore, to be able to predict how the hinge moment would vary as a function of the 3 parameters, namely the angle of attack, the deflection angle and the fractional length.

The resultant force of the pressure that acts only on the elevator can be calculated as follows:

$$C_{l,el} = \frac{Lift_{el}}{\frac{1}{2}\rho V_\infty^2 c} = 2\int_{\theta h}^{\pi}\left(A_0(1+\cos\theta) + \sum_{n=1}^{\infty} A_n.\sin n\theta.\sin\theta\right)d\theta$$

The moment of the distributed pressure, on the elevator only, about the leading edge is given by the following equation

$$C_{mLE,el} = -\int_{\theta h}^{\pi}\left(A_0(1+\cos\theta) + \sum_{n=1}^{\infty} A_n.\sin n\theta.\sin\theta\right)(1-\cos\theta).d\theta$$

$$C_{mLE,el} = -\left[\frac{A_0}{2}\int_{\theta h}^{\pi}(1-\cos 2\theta)\,d\theta + \sum_{n=1}^{\infty} A_n\int_{\theta h}^{\pi}\sin n\theta.\sin\theta.d\theta - \sum_{n=1}^{\infty}\frac{A_n}{2}\int_{\theta h}^{\pi}\sin n\theta.\sin 2\theta.d\theta\right]$$

The integrals can be evaluated as follows

$$\int_{\theta h}^{\pi}(1+\cos\theta).d\theta = \pi - \theta h - \sin\theta h$$

$$\int_{\theta h}^{\pi}(1-\cos 2\theta).d\theta = \pi - \theta h + \tfrac{1}{2}\sin 2\theta h$$

For n = 1 $$\int_{\theta h}^{\pi} \sin\theta.\sin\theta.d\theta = \tfrac{1}{2}\int_{\theta h}^{\pi}(1-\cos 2\theta).d\theta = \tfrac{1}{2}\left(\pi-\theta h+\tfrac{1}{2}\sin 2\theta h\right)$$

For $n \neq 1$ $$\int_{\theta h}^{\pi} \sin n\theta.\sin\theta.d\theta = \tfrac{1}{2}\int_{\theta h}^{\pi}\Big(\cos\big((n-1)\theta\big)-\cos\big((n+1)\theta\big)\Big).d\theta$$
$$= \tfrac{1}{2}\left(\frac{\sin\big((n+1)\theta h\big)}{n+1}-\frac{\sin\big((n-1)\theta h\big)}{n-1}\right)$$

For n = 1 $$\int_{\theta h}^{\pi} \sin 2\theta.\sin\theta.d\theta = \tfrac{1}{2}\left(\frac{\sin(3\theta h)}{3}-\frac{\sin(\theta h)}{1}\right)$$

For n = 2 $$\int_{\theta h}^{\pi} \sin 2\theta.\sin 2\theta.d\theta = \tfrac{1}{2}\int_{\theta h}^{\pi}(1-\cos 4\theta).d\theta = \tfrac{1}{2}\left(\pi-\theta h+\tfrac{1}{4}\sin 4\theta h\right)$$

For n>2 $$\int_{\theta h}^{\pi} \sin n\theta.\sin 2\theta.d\theta = \tfrac{1}{2}\int_{\theta h}^{\pi}\Big(\cos\big((n-2)\theta\big)-\cos\big((n+2)\theta\big)\Big).d\theta$$
$$= \tfrac{1}{2}\left(\frac{\sin\big((n+2)\theta h\big)}{n+2}-\frac{\sin\big((n-2)\theta h\big)}{n-2}\right)$$

Putting the above results together to evaluate the integrals, we can get the following

$$\sum_{n=1}^{\infty} A_n.\int_{\theta h}^{\pi} \sin n\theta.\sin\theta.d\theta = \tfrac{1}{2}A_1\left(\pi-\theta h+\tfrac{1}{2}\sin 2\theta h\right)$$
$$+\tfrac{1}{2}\sum_{n=2}^{\infty} A_n\left(\frac{\sin\big((n+1)\theta h\big)}{n+1}-\frac{\sin\big((n-1)\theta h\big)}{n-1}\right)$$

$$\sum_{n=2}^{\infty} A_n\left(\frac{\sin(n+1)\theta h}{n+1}-\frac{\sin(n-1)\theta h}{n-1}\right) = \frac{2\eta}{\pi}\sum_{n=2}^{\infty}\frac{\sin n\theta h}{n}.\left(\frac{\sin(n-1)\theta h}{n-1}-\frac{\sin(n+1)\theta h}{n+1}\right)$$
$$= \frac{\eta}{\pi}\sin\theta h.\sin 2\theta h = -\tfrac{1}{2}A_1\sin 2\theta h$$

Therefore
$$\sum_{n=1}^{\infty} A_n.\int_{\theta h}^{\pi} \sin n\theta.\sin\theta.d\theta = \tfrac{1}{2}\left[A_1\left(\pi-\theta h+\tfrac{1}{2}\sin 2\theta h\right)-\tfrac{1}{2}A_1\sin 2\theta h\right] = \tfrac{1}{2}A_1\left(\pi-\theta h\right)$$

For the coefficient of moment we need the following

$$\sum_{n=3}^{\infty} A_n\left(\frac{\sin(n+2)\theta h}{n+2}-\frac{\sin(n-2)\theta h}{n-2}\right) = \frac{2\eta}{\pi}\sum_{n=3}^{\infty}\frac{\sin n\theta h}{n}.\left(\frac{\sin(n-2)\theta h}{n-2}-\frac{\sin(n+2)\theta h}{n+2}\right)$$
$$= \frac{2\eta}{\pi}\left(\sin\theta h.\frac{\sin 3\theta h}{3}+\frac{\sin 2\theta h}{2}\frac{\sin 4\theta h}{4}\right)$$
$$= -A_1\frac{\sin 3\theta h}{3}-A_2\frac{\sin 4\theta h}{4}$$

Therefore

$$\sum_{n=1}^{\infty} A_n . \int_{\theta h}^{\pi} \sin n\theta . \sin 2\theta . d\theta = \tfrac{1}{2} A_1 \left(\frac{\sin 3\theta h}{3} - \frac{\sin \theta h}{1} \right) + \tfrac{1}{2} A_2 \left(\pi - \theta h + \frac{\sin 4\theta h}{4} \right)$$

$$- \tfrac{1}{2} A_1 \frac{\sin 3\theta h}{3} - \tfrac{1}{2} A_2 \frac{\sin 4\theta h}{4}$$

$$= - \tfrac{1}{2} A_1 \sin \theta h + \tfrac{1}{2} A_2 \left(\pi - \theta h \right)$$

The coefficient of moment of pressure distribution on the elevator, about the leading edge of the aerofoil is given as follows

$$C_{mLE,el} = -\frac{A_0}{2} \left(\pi - \theta h + \frac{\sin 2\theta h}{2} \right) - \frac{A_1}{2} \left(\pi - \theta h \right) - \left(\frac{A_1}{4} \sin \theta h - \frac{A_2}{4} \left(\pi - \theta h \right) \right)$$

$$C_{mLE,el} = -\frac{A_0}{2} \left(\pi - \theta h + \frac{\sin 2\theta h}{2} \right) - \frac{A_1}{2} \left(\pi - \theta h + \frac{\sin \theta h}{2} \right) + \frac{A_2}{4} \left(\pi - \theta h \right)$$

$$C_{mLE,el} = -\frac{1}{2} \left(\pi - \theta h + \frac{\sin 2\theta h}{2} \right) \left[\alpha - \left(1 - \frac{\theta h}{\pi} \right) . \eta \right]$$

$$- \frac{\sin \theta h}{\pi} \left(\pi - \theta h + \frac{\sin \theta h}{2} \right) . \eta + \frac{\sin 2\theta h}{4\pi} \left(\pi - \theta h \right) . \eta$$

$$C_{mLE,el} = -\frac{1}{2} \left(\pi - \theta h + \frac{\sin 2\theta h}{2} \right) \alpha - \frac{1}{2\pi} \left(\pi - \theta h \right)^2 . \eta - \frac{\sin \theta h}{\pi} \left(\pi - \theta h + \frac{\sin \theta h}{2} \right) . \eta$$

$$C_{mLE,el} = -\frac{\pi}{2} \left(1 - \frac{\theta h}{\pi} + \frac{\sin 2\theta h}{2\pi} \right) \alpha - \frac{\pi}{2} \left(\left(1 - \frac{\theta h}{\pi} \right)^2 + \frac{2 \sin \theta h}{\pi} \left(1 - \frac{\theta h}{\pi} + \frac{\sin \theta h}{2\pi} \right) \right) . \eta$$

$$C_{mLE,el} = -\frac{\pi}{2} \left(1 - \frac{\theta h}{\pi} + \frac{\sin 2\theta h}{2\pi} \right) \alpha - \frac{\pi}{2} \left(1 - \frac{\theta h}{\pi} + \frac{\sin \theta h}{\pi} \right)^2 . \eta \tag{175}$$

Let us now define the following

$$k_4 \left(\theta h \right) = 1 - \frac{\theta h}{\pi} - \frac{\sin \theta h}{\pi} = k_1 \left(\theta h \right) - \frac{\sin \theta h}{\pi} \tag{176}$$

$$k_5 \left(\theta h \right) = \left(1 - \frac{\theta h}{\pi} \right)^2 = k_1^2 \left(\theta h \right) \tag{177}$$

$$k_6 \left(\theta h \right) = 1 - \frac{\theta h}{\pi} + \frac{\sin 2\theta h}{\pi} = k_1 \left(\theta h \right) + \frac{\sin 2\theta h}{\pi} \tag{178}$$

$$k_7 \left(\theta h \right) = \left(1 - \frac{\theta h}{\pi} + \frac{\sin \theta h}{\pi} \right)^2 = k_3^2 \left(\theta h \right) \tag{179}$$

The lift coefficient and the leading edge moment coefficients of the elevator are now given by

$$C_{l,el} = 2\pi\left(k_4\left(\theta h\right).\alpha - k_5\left(\theta h\right).\eta\right) \tag{180}$$

$$C_{mLE,el} = -\frac{\pi}{2}\left(k_6\left(\theta h\right).\alpha - k_7\left(\theta h\right).\eta\right) \tag{181}$$

The hinge moment of the elevator is then given by the following

$$C_{mH,el} = C_{mLE,el} + \left(1-F\right).C_{l,el} \tag{182}$$

The lift and hinge moment coefficients of the elevator can also be rewritten as follows

$$C_{l,el}\left(\alpha,\eta\right) = a_1\left(F\right).\alpha - b_1\left(F\right).\eta \tag{183}$$

$$C_{mH,el}\left(\alpha,\eta\right) = -\left[a_2\left(F\right).\alpha - b_2\left(F\right).\eta\right] \tag{184}$$

where $a_1\left(F\right) = 2\pi.k_4\left(F\left(\theta h\right)\right)$ and $b_1\left(F\right) = 2\pi.k_5\left(F\left(\theta h\right)\right)$

$a_2\left(F\right) = \frac{\pi}{2}.\left[k_6\left(F\right) - 4\left(1-F\right)k_4\left(F\right)\right]$ and $b_2\left(F\right) = \frac{\pi}{2}.\left[k_7\left(F\right) - 4\left(1-F\right).k_5\right]$

The values of the above coefficients for some particular values of F are shown in the table below

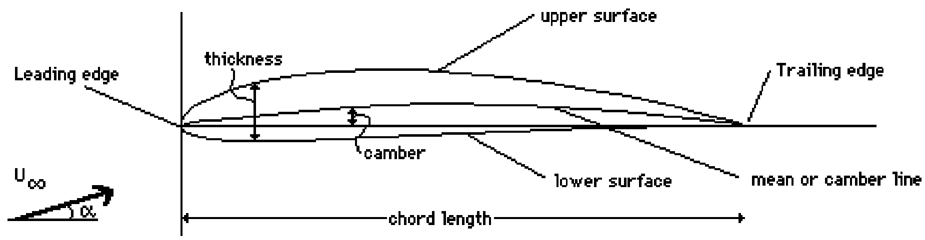
F	0.10	0.20	0.30	0.40	0.50
$a_1\left(F\right)$	0.08700	0.2546	0.4855	0.7793	1.1416
$b_1\left(F\right)$	0.2636	0.5474	0.8556	1.1939	1.5708
$a_2\left(F\right)$	- 0.2365	-0.2200	- 0.1268	0.0212	0.2146
$b_2\left(F\right)$	- 0.0088	0.0369	0.0869	0.2665	0.2665

It should be noted that the hinge moment coefficient (see equation (184)) is dependent on the angle of attack as well as on the angle of deflection. This is undesirable, since the hinge moment for a given deflection angle is different for different value of angle of attack. As far as the pilot is concerned, it is desirable that when s/he wants to change the setting of the angle of deflection, the force that s/he has to exert should be the same regardless of the angle of attack (which is not associated with trying to control the aircraft as such). However, for a particular value of F between 0.30 and 0.40 the value of $a_2\left(F\right)$ is zero. This means that for the particular value of F, the hinge moment is independent of angle of attack, thus that value of F should be chosen as the fractional length of the elevator (or the movable part of any control surface, such as the rudder on the vertical fin etc).

Additional information:

Airfoil Geometry
http://www.desktopaero.com/appliedaero/appliedaero.html

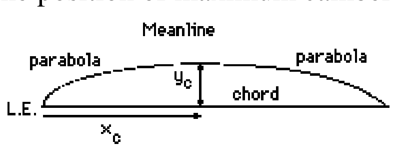
Airfoil geometry can be characterized by the coordinates of the upper and lower surface. It is often summarized by a few parameters such as: maximum thickness, maximum camber, position of max thickness, position of max camber, and nose radius. One can generate a reasonable airfoil section given these parameters. This was done by Eastman Jacobs in the early 1930's to create a family of airfoils known as the NACA Sections.



The NACA 4 digit and 5 digit airfoils were created by superimposing a simple meanline shape with a thickness distribution that was obtained by fitting a couple of popular airfoils of the time:

$y = \pm(t/0.2) * (.2969*x^{0.5} - .126*x - .3537*x^2 + .2843*x^3 - .1015*x^4)$

The camberline of 4-digit sections was defined as a parabola from the leading edge to the position of maximum camber, then another parabola back to the trailing edge.



NACA 4-Digit Series:

4	4	1 2
max camber	position	max thickness
in % chord	of max camber	in % of chord
	in 1/10 of c	

After the 4-digit sections came the 5-digit sections such as the famous NACA 23012. These sections had the same thickness distribution, but used a camberline with more curvature near the nose. A cubic was faired into a straight line for the 5-digit sections.

NACA 5-Digit Series:

2	3 0	1 2
approx max	position	max thickness
camber	of max camber	in % of chord
in % chord	in 2/100 of c	

The 6-series of NACA airfoils departed from this simply-defined family. These sections were generated from a more or less prescribed pressure distribution and were meant to achieve some laminar flow.

NACA 6-Digit Series:

6	3,	2	-	2	1 2
Six-Series	location of min Cp in 1/10 chord	half width of low drag bucket in 1/10 of Cl		ideal Cl in tenths	max thickness in % of chord

After the six-series sections, airfoil design became much more specialized for the particular application. Airfoils with good transonic performance, good maximum lift capability, very thick sections, very low drag sections are now designed for each use. Often a wing design begins with the definition of several airfoil sections and then the entire geometry is modified based on its 3-dimensional characteristics.
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The following web sites give the coordinates of various NACA aerofoils

Appendix I- Profiles http://www.pdas.com/profiles.htm
Appendix II- Mean Lines http://www.pdas.com/meanline.htm
Appendix III- 4 and 5 Digit Sections http://www.pdas.com/sections45.htm
Appendix III- 6-Series Sections http://www.pdas.com/sections6.htm
Appendix III- 6A-Series Sections http://www.pdas.com/sections6a.htm

See also
http://www.adl.gatech.edu/classes/lowspdaero/lospd5/lospd5.html

http://www.desktopaero.com/appliedaero/airfoils1/tatderivation.html

http://adg.stanford.edu/aa208/fundamentals/TATResults.html

Created by Hadi Winarto
Created on Monday, March 15, 2004
Modified: 18 March 2004
Last updated: 23 March 2004
Major revision 25 May 2004